

# On the vertical shear instability in magnetized protoplanetary discs

Can Cui <sup>1</sup>★ and Min-Kai Lin <sup>2,3</sup>

<sup>1</sup>*DAMTP, University of Cambridge, CMS, Wilberforce Road, Cambridge CB3 0WA, UK*

<sup>2</sup>*Institute of Astronomy and Astrophysics, Academia Sinica, Taipei 10617, Taiwan*

<sup>3</sup>*Physics Division, National Center for Theoretical Sciences, Taipei 10617, Taiwan*

Accepted 2021 May 24. Received 2021 May 23; in original form 2021 February 8

## ABSTRACT

The vertical shear instability (VSI) is a robust phenomenon in irradiated protoplanetary discs (PPDs). While there is extensive literature on the VSI in the hydrodynamic limit, PPDs are expected to be magnetized and their extremely low ionization fractions imply that non-ideal magnetohydrodynamic effects should be properly considered. To this end, we present linear analyses of the VSI in magnetized discs with Ohmic resistivity. We primarily consider toroidal magnetic fields, which are likely to dominate the field geometry in PPDs. We perform vertically global and radially local analyses to capture characteristic VSI modes with extended vertical structures. To focus on the effect of magnetism, a locally isothermal equation of state is employed. We find that magnetism provides a stabilizing effect to dampen the VSI, with surface modes, rather than body modes, being the first to vanish with increasing magnetization. Subdued VSI modes can be revived by Ohmic resistivity, where sufficient magnetic diffusion overcomes magnetic stabilization, and hydrodynamic results are recovered. We also briefly consider poloidal magnetic fields to account for the magnetorotational instability (MRI), which may develop towards surface layers in the outer regions of PPDs. The MRI grows efficiently at small radial wavenumbers, in contrast to the VSI. When resistivity is considered, the VSI dominates over the MRI for Ohmic Elsässer numbers  $\lesssim 0.09$  at plasma beta parameter  $\beta_Z \sim 10^4$ .

**Key words:** instabilities – MHD – methods: analytical – protoplanetary discs.

## 1 INTRODUCTION

It has been postulated over decades that the turbulence and angular momentum transport in most astrophysical accretion discs are mediated by the magnetorotational instability (MRI; Chandrasekhar 1961; Balbus & Hawley 1991). However, protoplanetary discs (PPDs) are distinguished by their extremely weakly ionized gas (Gammie 1996; Armitage 2011), where gas and magnetic fields are poorly coupled, and the MRI turbulence is either quenched or dampened in the bulk of the disc (Bai & Stone 2011; Perez-Becker & Chiang 2011; Simon et al. 2013a,b). Instead, angular momentum transport is dominated by magnetized disc winds, leaving the bulk disc mostly laminar (Bai & Stone 2013; Gressel et al. 2015, 2020; Bai et al. 2016; Bai 2017; Béthune, Lesur & Ferreira 2017).

Nevertheless, some level of turbulence is expected in PPDs to account for the recent Atacama Large Millimeter/submillimeter Array (ALMA) observations of molecular line emissions (Teague et al. 2016; Flaherty et al. 2017, 2018, 2020). Furthermore, turbulence may serve as an essential ingredient in many stages of planet formation. Turbulence affects the gravitational sedimentation (Dubrulle, Morfill & Sterzik 1995; Johansen & Klahr 2005; Youdin & Lithwick 2007), radial diffusion (Clarke & Pringle 1988), and collisional growth (Ormel & Cuzzi 2007; Birmstiel, Dullemond & Brauer 2010) of dust particles. Long-lived vortices induced by turbulence (e.g. Raettig, Klahr & Lyra 2015; Manger & Klahr 2018) can concentrate dust particles (Barge & Sommeria

1995; Klahr & Henning 1997; Cuzzi, Hogan & Shariff 2008) and seed planetesimal formation through streaming instability or gravitational instability (Youdin & Goodman 2005; Johansen et al. 2007; Chiang & Youdin 2010), whereas the growth of streaming instability can be substantially diminished by a moderate level of turbulent viscosity (Chen & Lin 2020; Umurhan, Estrada & Cuzzi 2020). Turbulence also influences the radial migration of planets, as well as the flow morphology and gap formation around them (Nelson & Papaloizou 2004; Papaloizou, Nelson & Snellgrove 2004). Therefore, understanding the origin and characteristics of turbulence in PPDs is essential to many aspects of planet formation and evolution.

The lack of magnetohydrodynamic (MHD) turbulence in a PPD led to a surge in the interest of purely hydrodynamic instabilities (Lyra & Umurhan 2019; Weiss, Bai & Fu 2021). Among the most explored are the vertical shear instability (VSI; Nelson, Gressel & Umurhan 2013, hereafter N13; Lin & Youdin 2015; Latter & Papaloizou 2018, hereafter LP18; Cui & Bai 2020), the convective overstability in its linear (Klahr & Hubbard 2014; Lyra 2014; Latter 2016) and non-linear (subcritical baroclinic instability; Klahr & Bodenheimer 2003; Petersen, Julien & Stewart 2007a; Petersen, Stewart & Julien 2007b; Lesur & Papaloizou 2010) phases, and the zombie vortex instability (Marcus et al. 2013, 2015; Lesur & Latter 2016; Umurhan, Shariff & Cuzzi 2016b). These instabilities set in under certain thermodynamic and structural conditions, thereby operating at distinct regions of PPDs (Malygin et al. 2017; Lyra & Umurhan 2019; Pfeil & Klahr 2019). The VSI is of particular interest as it extends a large portion of the disc (e.g. Lin & Youdin 2015; Lyra & Umurhan 2019).

\* E-mail: [cc795@cam.ac.uk](mailto:cc795@cam.ac.uk)

The VSI is inherited from the Goldreich–Schubert–Fricke instability (Goldreich & Schubert 1967; Fricke 1968) and is initially discovered in the context of differentially rotating stars. Its importance to accretion discs was later explored by Urpin & Brandenburg (1998), Urpin (2003), and Arlt & Urpin (2004). The applicability of the VSI to PPDs has been demonstrated only recently in N13. It quickly drew intensive interests (e.g. Stoll & Kley 2014; Barker & Latter 2015; Umurhan, Nelson & Gressel 2016a; LP18; Lin 2019; Cui & Bai 2020; Flock et al. 2020; Schäfer, Johansen & Banerjee 2020; Flores-Rivera et al. 2020) and is considered to be a promising hydrodynamic mechanism in driving turbulence in PPDs.

A differentially rotating disc with Keplerian profile is stable according to the Rayleigh criterion (Chandrasekhar 1961). The presence of vertical shear can destabilize inertial waves in a vertically global disc model (Barker & Latter 2015). Nevertheless, a fluid element also experiences stabilizing effects from vertical buoyancy, which impedes the VSI growth. This can be overcome by sufficiently rapid cooling that brings the perturbed fluid element to reach local thermal equilibrium with its surroundings, hence diminishing the buoyancy. Local linear analyses demonstrate that the unstable modes are characterized by short radial wavelengths and maximum growth rates much smaller than the orbital frequency (Urpin 2003; N13). Two classes of VSI modes have been identified: rapidly growing surface modes concentrated near the disc surface and more vertically extended body modes (N13; Barker & Latter 2015; Lin & Youdin 2015). The body modes can be further categorized into breathing and corrugation modes, depending on the symmetry about the mid-plane (N13).

The non-linear evolution of the VSI has been examined by hydrodynamic simulations. In accordance with the linear theory, the VSI is triggered when thermal relaxation time-scales are less than 0.01–0.1 times the local dynamical time-scales, and the wave modes exhibit elongated vertical wavelengths (N13). While the surface modes possess the fastest growth rate, the body (corrugation) modes eventually take over, dominating the non-linear evolution. Fully developed VSI turbulence yields a Shakura–Sunyaev (Shakura & Sunyaev 2009)  $\alpha$  value on the order of  $10^{-4}$ – $10^{-3}$  (N13; Stoll & Kley 2014; Cui & Bai 2020). Non-axisymmetric 3D simulations show the development of vortices (Richard, Nelson & Umurhan 2016; Manger & Klahr 2018; Flock et al. 2020; Manger et al. 2020; Pfeil & Klahr 2020). Incorporating dust particles, numerical simulations show that VSI can stir up dust grains against vertical settling, but may also concentrate them through inducing dust-trapping vortices (Stoll & Kley 2016; Flock et al. 2017; Lin 2019; Schäfer et al. 2020).

Most studies of the VSI to date have neglected magnetic fields despite their importance in the evolution of PPDs. Two recent works extend analyses of the VSI to MHD regimes. Local linear stability analyses in the ideal MHD limit show that weak magnetic fields are favoured to excite the VSI, specifically when plasma beta  $\beta \gtrsim 400$  for thin discs (LP18), where  $\beta$  is the ratio of gas to magnetic pressure. They also find the MRI growth rates exceed that for VSI modes, but the wave vectors of the two are perpendicular to each other. For resistive discs, LP18 estimate a critical Ohmic Elsässer number of  $\sim 1/h$ , where  $h$  is the disc aspect ratio, for the VSI to operate, by requiring the magnetic diffusion time-scale to be shorter than the Alfvén wave propagation time-scales. Non-linear MHD simulations, applicable to outer regions of the disc, demonstrate that the VSI can initiate and sustain turbulence in magnetized discs (Cui & Bai 2020). Weak ambipolar diffusion strength, or the enhanced coupling between gas and magnetic fields, works as stabilizing effects to dampen the VSI growth.

Previous hydrodynamic models suggest that effective VSI growths span over  $\sim 5$ – $100$  au in PPDs (Lin & Youdin 2015; Pfeil & Klahr 2019). These regions are susceptible to all three non-ideal MHD effects – Ohmic resistivity, Hall effect, and ambipolar diffusion (Wardle 2007; Bai 2011). However, a quantitative analysis in vertically global, magnetized discs with non-ideal effects is still lacking. Such analyses can be useful for understanding the VSI mode properties in real PPDs and interpreting non-linear simulations. Hence, in this work, we extend the linear stability analysis of the VSI to weakly ionized gas in a vertically global disc model. We remark that a vertically global analysis is necessary for a proper description of elongated body modes of the VSI, which have been found to dominate in numerical simulations (N13; Stoll & Kley 2014; Cui & Bai 2020). We consider ideal MHD and further include Ohmic resistivity as a proxy for non-ideal effects. We focus primarily on the effect of toroidal magnetic fields. However, in a local model, we also investigate the dominance between MRI and VSI by considering purely poloidal magnetic fields.

The plan of the paper is as follows. In Section 2, we introduce the basic formulation and establish the equilibrium state of the problem. In Section 3, we derive and discuss the Solberg–Hoiland stability criteria for magnetized discs. In Section 4, we present the linearized equations and detail the analytical and numerical methods used, with results shown in Section 5, for a purely toroidal background magnetic field. We also conduct a brief analysis for a purely poloidal background magnetic field and compare the MRI growth rates with the VSI in Section 6. Finally, we discuss the results in Section 7 and summarize our main findings in Section 8.

## 2 BASIC EQUATIONS AND EQUILIBRIA

Consider a gaseous, inviscid, magnetized PPD. The gas density, velocity, and magnetic field are denoted by  $\rho$ ,  $\mathbf{v}$ ,  $\mathbf{B}$ , respectively. Our formulations are presented in cylindrical  $(R, \phi, Z)$  coordinates centred on the protostar, although the spherical radius  $r$  is also used to simplify expressions. The basic dynamical equations written in SI units are

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla \Pi - \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} + \rho \nabla \Phi = 0, \quad (2)$$

where  $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$  is the material derivative and  $\mu_0$  is the magnetic permeability. The gravitational potential is given by

$$\Phi = -\frac{GM}{r}, \quad (3)$$

where  $G$  is gravitational constant, and  $M$  is the mass of the central star. The total pressure

$$\Pi = P + P_B \quad (4)$$

is the sum of thermal pressure  $P$  and magnetic pressure

$$P_B = \frac{B^2}{2\mu_0}, \quad (5)$$

where  $B = |\mathbf{B}|$ . The strength of the magnetic field is parametrized by the ratio of gas pressure to magnetic pressure,

$$\beta = \frac{P}{P_B}. \quad (6)$$

## 2.1 Induction equation

The evolution of magnetic fields is governed by the induction equation, in which three non-ideal MHD effects manifest,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \mathbf{v} \times \mathbf{B} - \frac{\mathbf{J}}{\sigma_B} - \frac{\mathbf{J} \times \mathbf{B}}{en_e} + \frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{\gamma_i \rho \rho_i} \right]. \quad (7)$$

Here,  $\sigma_B$  is resistivity,  $e$  is electric charge, and  $n_e$  is electron number density. The drag coefficient  $\gamma_i$  represents the momentum transfer between ion and neutral collisions, and the ion density is denoted by  $\rho_i$ . On the right-hand side, the four terms each corresponds to the standard inductive term, Ohmic resistivity, the Hall effect, and ambipolar diffusion. Ohmic resistivity is relevant to electron–neutral collisions. Ambipolar diffusion corresponds to ion–neutral drift. The Hall effect is associated with ion–electron drift, and it differs fundamentally from Ohmic resistivity and ambipolar diffusion as being a non-dissipative process. The relation between the current density  $\mathbf{J}$  and the magnetic field is completed by the Maxwell equation,

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}. \quad (8)$$

The displacement current in equation (8) is self-consistently neglected for non-relativistic MHD (Balbus 2009). Finally, the magnetic field also satisfies the solenoidal condition,

$$\nabla \cdot \mathbf{B} = 0. \quad (9)$$

To characterize the non-ideal MHD effects, it is convenient to define the diffusion coefficients

$$\eta_O = \frac{1}{\mu_0 \sigma_B}, \quad \eta_H = \frac{B}{\mu_0 en_e}, \quad \eta_A = \frac{B^2}{\mu_0 \gamma_i \rho \rho_i} \quad (10)$$

for Ohmic resistivity, the Hall effect, and ambipolar diffusion. At a given ionization fraction, the Ohmic diffusivity  $\eta_O$  is independent of field strength and density, whereas the Hall diffusivity  $\eta_H \propto B/\rho$ , and the ambipolar diffusivity  $\eta_A \propto B^2/\rho^2$ . Hence, ambipolar diffusion dominates in regions of strong fields or low densities, Ohmic resistivity dominates in regions of weak fields or high densities, and the Hall effect governs in between (Lesur 2020). We further introduce dimensionless Elsässer numbers,

$$\Lambda = \frac{v_A^2}{\eta_O \Omega_K}, \quad \chi = \frac{v_A^2}{\eta_H \Omega_K}, \quad \text{Am} = \frac{v_A^2}{\eta_A \Omega_K}, \quad (11)$$

where the Alfvén velocity  $v_A$  is given by

$$v_A^2 = \frac{|\mathbf{B}|^2}{\mu_0 \rho}. \quad (12)$$

### 2.1.1 Toroidal fields in axisymmetric discs

Toroidal magnetic fields have been shown to dominate in global non-ideal MHD simulations, with the saturation of VSI turbulence (e.g. Bai 2017; Béthune et al. 2017; Cui & Bai 2020). Furthermore, the VSI is an axisymmetric instability (N13). These results motivate us to primarily consider purely toroidal fields in axisymmetric discs. In this case, the Hall effect vanishes, and we show here that ambipolar diffusion behaves the same way as Ohmic resistivity. In the limit of  $\mathbf{B} = (0, B_\phi, 0)$ , we expand the numerator of the last term on the right-hand side of equation (7), corresponding to ambipolar diffusion,

$$(\mathbf{J} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{J} \cdot \mathbf{B})\mathbf{B} - B^2 \mathbf{J}. \quad (13)$$

This implies that for electric currents being everywhere perpendicular to magnetic fields, as the case for purely toroidal field geometry,

ambipolar diffusion can be treated as an effective resistivity with field dependency (Balbus & Terquem 2001). Consequently, the induction equation reduces to

$$\frac{dB_\phi}{dt} = -(\nabla \cdot \mathbf{v})B_\phi + \frac{B_\phi v_R}{R} + \eta \left( \nabla^2 B_\phi - \frac{B_\phi}{R^2} \right), \quad (14)$$

where  $\eta = \eta_O + \eta_A$ , and we assume a constant  $\eta$  throughout. The second and final terms on the right-hand side result from the curvilinear geometry.

## 2.2 Effective energy equation

The ideal gas law is given by

$$P = \frac{\mathcal{R}}{\mu} \rho T, \quad (15)$$

where  $\mathcal{R}$  is the gas constant,  $\mu$  is the mean molecular weight, and  $T$  is the gas temperature. We define the isothermal sound speed  $c_s$ , pressure scale height  $H$ , and disc aspect ratio  $h$  through

$$c_s^2 \equiv \frac{P}{\rho}, \quad H \equiv \frac{c_s}{\Omega_K}, \quad h \equiv \frac{H}{R}, \quad (16)$$

where  $\Omega_K = \sqrt{GM/R^3}$  is the Keplerian angular velocity. In the hydrodynamic limit, isothermality is most favourable for the VSI because the stabilizing effect stabilizing from gas buoyancy is absent (N13; Lin & Youdin 2015). To focus on the effect of magnetic fields, we adopt a locally isothermal thermodynamic response, which is applicable to the outer parts of PPDs wherein the temperature is regulated by stellar irradiation (Chiang & Goldreich 1997). Thereby, the energy equation of gas pressure becomes

$$\frac{dP}{dt} + P \nabla \cdot \mathbf{v} = \rho \mathbf{v} \cdot \nabla c_s^2. \quad (17)$$

Consider the evolution of the magnetic pressure associated with a purely azimuthal field,

$$\frac{dP_B}{dt} = \frac{B_\phi}{\mu_0} \frac{dB_\phi}{dt}. \quad (18)$$

Combining equations (14), (17), and (18), we formulate an effective energy equation

$$\begin{aligned} \frac{d\Pi}{dt} + \gamma_B \Pi (\nabla \cdot \mathbf{v}) &= \rho \mathbf{v} \cdot \nabla c_s^2 + \frac{B_\phi^2}{\mu_0 R} v_R \\ &+ \frac{B_\phi}{\mu_0} \eta \left( \nabla^2 B_\phi - \frac{B_\phi}{R^2} \right), \end{aligned} \quad (19)$$

where the effective adiabatic index is defined by

$$\gamma_B \equiv \frac{\beta + 2}{\beta + 1}. \quad (20)$$

Equation (19) resembles the energy equation for a fluid subject to heating or cooling in the hydrodynamic limit (Lin & Youdin 2015). It shows that an axisymmetric, locally isothermal, magnetized gas with a purely azimuthal field behaves like an unmagnetized gas with adiabatic index  $\gamma_B$ ,<sup>1</sup> while it is also subject to non-adiabatic effects on the right-hand side: the locally isothermal thermodynamic response, magnetic diffusion, and curvature effects. Note that  $\gamma_B(\beta)$  here is not necessarily constant because  $\beta(R, Z, t)$  can be non-uniform and

<sup>1</sup>We remark that this analogy can be generalized to locally polytropic discs where  $P = K \rho^\Gamma$ , where  $K$  is a prescribed function of position and  $\Gamma$  is the polytropic index. In this case the effective adiabatic index becomes  $\gamma_B = (\Gamma \beta + 2)/(\beta + 1)$ .

can evolve in time. For  $\beta \rightarrow \infty$ , we recover a locally isothermal gas with unit adiabatic index  $\gamma_B \rightarrow 1$ . On the other hand, a strongly magnetized disc with  $\beta \rightarrow 0$  is equivalent to a fluid with an effective adiabatic index  $\gamma_B \rightarrow 2$ .

### 2.3 Equilibrium state

We consider axisymmetric steady states with a purely azimuthal velocity and magnetic field. The mid-plane gas density and temperature are prescribed as

$$\rho_0(R) = \rho_0(R_0) \left( \frac{R}{R_0} \right)^{-q_D}, \quad (21)$$

$$T(R) = T(R_0) \left( \frac{R}{R_0} \right)^{-q_T}, \quad (22)$$

where  $R_0$  is a reference radius, and  $q_T$  and  $q_D$  are constant power-law indices. The equilibrium solutions satisfy the equation of motion,

$$\frac{\partial \Phi}{\partial R} + \frac{1}{\rho} \frac{\partial \Pi}{\partial R} + \frac{2P_B}{\rho R} = R\Omega^2, \quad (23)$$

$$\frac{\partial \Phi}{\partial Z} + \frac{1}{\rho} \frac{\partial \Pi}{\partial Z} = 0, \quad (24)$$

where  $\Omega \equiv v_\phi/R$  is the angular frequency. These equations may be solved explicitly for a given magnetic field configuration. We consider two special cases in the following, depending on whether resistivity is included.

#### 2.3.1 Constant- $\beta$ discs

We consider constant- $\beta$  discs for ideal MHD ( $\eta = 0$ ). The effective energy equation (19) is then satisfied identically so the solutions below represent exact equilibria. In this case, the vertical gradient of equilibrium angular velocity is

$$\frac{\partial \Omega^2}{\partial Z} = -\frac{q_T}{R^2} \frac{GMZ}{r^3}. \quad (25)$$

The density and rotation profiles are

$$\rho(R, Z) = \rho_0(R) \exp \left[ h^{-2} \left( \frac{R}{r} - 1 \right) \frac{\beta}{1 + \beta} \right], \quad (26)$$

$$\Omega^2(R, Z) = \Omega_K^2 \left[ (1 - q_T) - \frac{1 + \beta}{\beta} \left( q_T + q_D + \frac{2}{1 + \beta} \right) h^2 + \frac{q_T R}{r} \right]. \quad (27)$$

As  $\beta \rightarrow \infty$ , we recover the hydrodynamic limit (N13). The thin disc approximation ( $h \ll 1$ ) leads to  $\Omega \approx \Omega_K$ . As  $\beta \rightarrow 0$ , the density gradient profile shows that a strongly magnetized disc becomes vertically unstratified. A small  $\beta \leq 1$  leads to strong deviations from the Keplerian rotation, but a minimum value of  $\beta$  is required to ensure  $\Omega^2 > 0$  at the disc mid-plane,

$$\beta > \frac{(q_T + q_D - 2)h^2}{1 - (q_T + q_D)h^2}. \quad (28)$$

For typical PPD parameters,  $h \sim 0.05$  and  $q_D, q_T$  are of order unity, so the above requirement becomes  $\beta \gtrsim O(10^{-3})$ . This is easily satisfied for the weakly magnetized discs with  $\beta \gtrsim 1$  that we consider.

#### 2.3.2 Constant- $B_\phi$ discs

We consider constant- $B_\phi$  discs for ideal MHD and resistive discs ( $\eta \neq 0$ ). Resistive discs present approximate equilibrium solutions because there is a slow diffusion of the magnetic field due to the global curvature term in equation (19). However, this is not expected

to significantly affect radially localized dynamics such as the VSI. In this case,  $\beta$  is no longer a constant but declines with height,

$$\beta(R, Z) = \beta_0(R) \frac{\rho(R, Z)}{\rho_0(R)}, \quad (29)$$

where  $\beta_0 = \beta(R, 0)$ . The disc becomes more strongly magnetized with increasing height.

For constant- $B_\phi$  discs, magnetic pressure does not contribute to the vertical equilibrium, and we recover the hydrodynamic limit for density profile,

$$\rho(R, Z) = \rho_0(R) \exp[h^{-2}(R/r - 1)]. \quad (30)$$

The vertical shear gradient differs from constant- $\beta$  discs because the curvature term in equation (23) now depends on  $Z$  through  $\rho$ . We find

$$\frac{\partial \Omega^2}{\partial Z} = \left( \frac{2}{\beta} - q_T \right) \frac{1}{R^2} \frac{GMZ}{r^3}, \quad (31)$$

and the rotation profile is

$$\Omega^2(R, Z) = \Omega_K^2 \left[ (1 - q_T) - \left( q_T + q_D + \frac{2}{\beta} \right) h^2 + \frac{q_T R}{r} \right]. \quad (32)$$

Accordingly, even a strictly isothermal disc ( $q_T = 0$ ) can exhibit vertical shear due to the curvature term. Thus, we may expect a corresponding VSI growth, but this should be examined further in radially global models. For PPDs where magnetic fields are weak, the angular velocity profiles in equations (27) and (32) converge to the hydrodynamic limit.

## 3 SOLBERG-HOILAND CRITERIA

The Solberg–Hoiland criteria describe the linear hydrodynamic stability of ideal fluids against axisymmetric and adiabatic perturbations (Tassoul 1978). Now, our axisymmetric, magnetized discs with purely azimuthal fields obey a similar energy equation (19) as in hydrodynamics. If, in addition, curvature terms and magnetic tension forces can be neglected, then our disc models satisfy the same form of equations as in adiabatic hydrodynamics. This motivates us to formulate an equivalent Solberg–Hoiland stability criterion for magnetized discs as follows.

The standard hydrodynamic Solberg–Hoiland criteria are expressed in terms of the gradient of pressure  $P$ , entropy  $S \propto \ln(P^{1/\gamma}/\rho)$ , and angular frequency  $\Omega$ , where  $\gamma$  is the adiabatic index. By setting  $P \rightarrow \Pi$  and  $\gamma \rightarrow \gamma_B$ , we find that the stability is ensured in a magnetized disc if

$$\kappa^2 - \frac{1}{\rho} \nabla \Pi \cdot \nabla S_B > 0, \quad (33)$$

$$- \frac{1}{\rho} \frac{\partial \Pi}{\partial Z} \left( \kappa^2 \frac{\partial S_B}{\partial Z} - R \frac{\partial \Omega^2}{\partial Z} \frac{\partial S_B}{\partial R} \right) > 0, \quad (34)$$

where

$$\kappa^2 \equiv \frac{1}{R^3} \frac{\partial (R^4 \Omega^2)}{\partial R}$$

is the square of epicyclic frequency, and

$$\nabla S_B \equiv \frac{1}{\gamma_B} \nabla \ln \Pi - \nabla \ln \rho \quad (35)$$

defines the gradient of effective entropy  $S_B$  of a locally isothermal fluid with an azimuthal field. Correspondingly, we define

$$N_R^2 = -\frac{1}{\rho} \frac{\partial \Pi}{\partial R} \frac{\partial S_B}{\partial R}, \quad N_Z^2 = -\frac{1}{\rho} \frac{\partial \Pi}{\partial Z} \frac{\partial S_B}{\partial Z} \quad (36)$$



as the square of buoyancy frequencies in the radial and vertical directions, respectively.

To examine whether magnetized disc models are stable subject to the modified Solberg–Hoiland criteria, we first derive the vertical buoyancy frequency for constant- $\beta$  and constant- $B_\phi$  thin discs,

$$N_Z^2 = \frac{\beta}{(\beta + 2)(\beta + 1)} \frac{Z^2}{H^2} \Omega_K^2 \quad (\text{constant-}\beta), \quad (37)$$

$$N_Z^2 = \frac{2}{\beta(R, Z) + 2} \frac{Z^2}{H^2} \Omega_K^2 \quad (\text{constant-}B_\phi). \quad (38)$$

As  $\beta \rightarrow \infty$ ,  $N_Z^2 \propto 1/\beta \rightarrow 0$  for both models, and we recover a locally isothermal hydrodynamic disc with vanishing vertical buoyancy. As  $\beta \rightarrow 0$ , for constant- $\beta$  discs,  $N_Z^2 \propto \beta$  and the buoyancy effect vanishes since there is no vertical density stratification in this limit. In a constant- $B_\phi$  disc,  $\beta \rightarrow 0$  occurs for large  $|Z|$  as in equation (29), so that  $N_Z^2 \rightarrow \Omega_K^2 Z^2 / H^2$ , and the disc expected to be strongly stabilized by magnetic buoyancy.

We now examine whether the first Solberg–Hoiland criterion is satisfied. Equation (33) can be cast into

$$\kappa^2 + N_R^2 + N_Z^2 > 0. \quad (39)$$

A stable stratification,  $N_{R,Z}^2 > 0$ , requires the effective entropy to increase with  $R$  or  $Z$  because total pressure gradients are usually negative. In thin weakly magnetized discs,  $|N_R| \sim O(\Omega_K h) \ll \kappa$  and  $N_Z^2 > 0$  (equations 37 and 38), thus the first criterion is generally satisfied.

Next, we examine the second Solberg–Hoiland criterion. Recall that the Solberg–Hoiland criteria apply to adiabatic flows. Thus, equations (33) and (34) should only be applied if the governing equations of the magnetized disc (equations 1–2, and 19) can map exactly to adiabatic hydrodynamics. That is, the right-hand side of equation (19) and the magnetic tension force in equation (2) should be negligible. This requires

- (i) a strictly isothermal disc (constant  $c_s$  in  $R, Z$ );
- (ii) ideal MHD ( $\eta = 0$ );
- (iii) weak magnetic fields ( $\beta \gg 1$ ) so that curvature terms associated with magnetic fields can be neglected (Pessah & Psaltis 2005).

The first restriction implies  $\partial\Omega/\partial Z = 0$ . The second criterion then becomes

$$\kappa^2 N_Z^2 > 0, \quad (40)$$

which is satisfied in both of our disc models. To study stability with vertical shear, which requires a radially varying disc temperature and non-ideal MHD effects, we must solve the linearized equations explicitly as conducted in the following section.

## 4 LINEAR PROBLEM

### 4.1 Perturbation equations

We consider axisymmetric Eulerian perturbations for  $\mathbf{v}'$ ,  $\rho'$ , and  $\Pi'$  of the form

$$\rho' \propto \exp(\sigma t + ik_R R). \quad (41)$$

The complex frequency is denoted by  $\sigma = s + i\omega$ , and a real radial wavenumber  $k_R$  is taken. We assume radially localized disturbances,  $k_R R \gg 1$ . The background disc variables are evaluated at the reference radius  $R_0$ , but their vertical dependence is retained. Curvature terms resulting from the cylindrical geometry are neglected, which restricts

the analyses to weak magnetic fields (Pessah & Psaltis 2005). By assuming  $\mathbf{B}' = (0, B'_\phi, 0)$ , the linearized perturbation equations read

$$\sigma v'_R - 2\Omega v'_\phi - \frac{\rho'}{\rho^2} \frac{\partial \Pi}{\partial R} + ik_R \frac{\Pi'}{\rho} = 0, \quad (42)$$

$$\sigma v'_\phi + \frac{\kappa^2}{2\Omega} v'_R + \frac{\partial v_\phi}{\partial Z} v'_Z = 0, \quad (43)$$

$$\sigma v'_Z - \frac{\rho'}{\rho^2} \frac{\partial \Pi}{\partial Z} + \frac{1}{\rho} \frac{\partial \Pi'}{\partial Z} = 0, \quad (44)$$

$$\sigma \rho' + \left( ik_R \rho + \frac{\partial \rho}{\partial R} \right) v'_R + \rho Z \frac{\partial \rho}{\partial Z} v'_Z + \rho \frac{\partial v'_Z}{\partial Z} = 0, \quad (45)$$

$$\sigma \Pi' + \left( ik_R \Pi \gamma_B + \frac{\partial \Pi}{\partial R} - \rho \frac{\partial c_s^2}{\partial R} \right) v'_R + \frac{\partial \Pi}{\partial Z} v'_Z + \gamma_B \Pi \frac{\partial v'_Z}{\partial Z} + \frac{B_\phi}{\mu_0} \eta \nabla^2 B'_\phi = 0. \quad (46)$$

Note that the magnetic tension force vanishes for a purely toroidal field and neglecting curvature terms. The perturbed azimuthal magnetic field  $B'_\phi$  can be expressed as

$$B'_\phi = \frac{\mu_0}{B_\phi} (\Pi' - \rho' c_s^2). \quad (47)$$

When  $\beta \rightarrow \infty$  and hence  $\Pi' \rightarrow P'$ , the set of linearized equations recovers the hydrodynamic limit (Lin & Youdin 2015). For finite magnetic field strengths and  $\eta = 0$ , these equations describe the ideal MHD limit. For  $\eta > 0$ , the set of equations describes non-ideal MHD.

### 4.2 Analytical solutions

Analytic solutions can be obtained in the limit of ideal MHD ( $\eta = 0$ ) with a large and constant  $\beta$ , by solving the linearized equations with polynomial solutions. To do so, we make the following simplifying assumptions (Lin & Youdin 2015).

(i) A fully radially local approximation ( $\partial/\partial R = 0$ ) for background discs. However, the vertical shear that originates from the radial temperature profile ( $\partial T/\partial R$ ) is retained.

(ii) We set  $\Omega = \Omega_K$  and  $\kappa = \Omega_K$  where they appear explicitly and without vertical derivatives. As seen in equation (27), the Keplerian approximation is only valid for large  $\beta$ . A strong magnetic field will lead to substantial deviation from  $\Omega_K$ .

(iii) In the hydrodynamic limit, the VSI is an overstability due to the destabilization of inertial waves (Barker & Latter 2015). We expect a similar result for weak magnetizations and consider low frequency modes with  $|\sigma| \ll \Omega_K$  to filter out acoustic waves (Lubow & Pringle 1993).

(iv) We consider thin discs and set

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \Pi}{\partial Z} &= -\Omega_K^2 Z, \\ \frac{\partial \ln \rho}{\partial Z} &= -\frac{\beta}{1 + \beta} \frac{Z}{H^2}, \\ R \frac{\partial \Omega^2}{\partial Z} &= -q_T \frac{Z}{R} \Omega_K^2. \end{aligned} \quad (48)$$

To non-dimensionalize the perturbation equations, we choose the appropriate scalings for time-scales, length-scales, and velocities to be the Keplerian orbital time  $\Omega_K^{-1}$ , background sound speed  $c_s$ , and pressure scale-height  $H$ . We thus write

$$\sigma \rightarrow \Omega_K \sigma^*, \quad Z \rightarrow H Z^*, \quad k_R \rightarrow K/H, \quad (49)$$

where scaled variables are denoted by an asterisk and are omitted below. Equations (42) – (46) can be combined into a single second-order ordinary differential equation,

$$\frac{d^2 v'_Z}{dZ^2} - a_1 \frac{dv'_Z}{dZ} Z + v'_Z (a_2 - a_3 Z^2) = 0, \quad (50)$$

where  $a_1$ ,  $a_2$ , and  $a_3$  are constants in  $Z$  and are defined by

$$\begin{aligned} a_1 &= \frac{\beta}{1 + \beta} - iKq_T h, \\ a_2 &= -\frac{\beta}{2 + \beta} - \sigma^2 K^2 + iKq_T h, \\ a_3 &= \frac{\beta}{2 + \beta} (\gamma_B - 1) (K^2 + iKq_T h). \end{aligned} \quad (51)$$

Equation (50) is equivalent to the hydrodynamic result derived by Lin & Youdin (2015, see their equation 29) without cooling and assuming  $K \gg 1$ . This can be seen by setting  $Z \equiv \tilde{Z} \sqrt{(1 + \beta)/\beta}$ ,  $K \equiv \tilde{K} \sqrt{\beta/(1 + \beta)}$ , and  $h \equiv \tilde{h} \sqrt{\beta/(1 + \beta)}$ . However, they also show that in the hydrodynamic limit, neglecting the radial structure of disc while retaining vertical shear is only valid for gas that is nearly isothermal in its thermodynamic response. We thus expect equation (50) to only apply for weak magnetizations where  $\gamma_B \rightarrow 1$ . In the limit  $\beta \rightarrow \infty$  and hence  $a_3 \rightarrow 0$ , we recover the equation for the VSI in locally isothermal discs (Barker & Latter 2015; Lin & Youdin 2015). When there is no vertical shear ( $q_T = 0$ ), equation (50) is analogous to that obtained by Lubow & Pringle (1993) for axisymmetric waves in adiabatic discs.

We bring equation (50) into a form that can be solved analytically (Lubow & Pringle 1993). Define a variable  $Y(Z)$ ,

$$Y(Z) = v'_Z(Z) \exp(-Z^2 \xi / 2), \quad (52)$$

where  $\xi$  is a constant to be determined. Plugging this into equation (50) yields an ordinary differential equation in  $Y$ ,

$$\frac{d^2 Y}{dZ^2} - (a_1 - 2\xi)Z \frac{dY}{dZ} + Y[(a_2 + \xi) + (\xi^2 - a_1 \xi - a_3)Z^2] = 0. \quad (53)$$

Then choose  $\xi$  to eliminate the  $Z^2$  term,

$$\xi = \frac{1}{2}a_1 \pm \frac{1}{2}\sqrt{a_1^2 + 4a_3}, \quad (54)$$

which brings equation (53) into

$$\frac{d^2 Y}{dZ^2} - (a_1 - 2\xi)Z \frac{dY}{dZ} + Y(a_2 + \xi) = 0. \quad (55)$$

We proceed to represent solutions of  $Y$  by  $Y_n(Z)$ , an  $n$ th order polynomial in  $Z$ ,

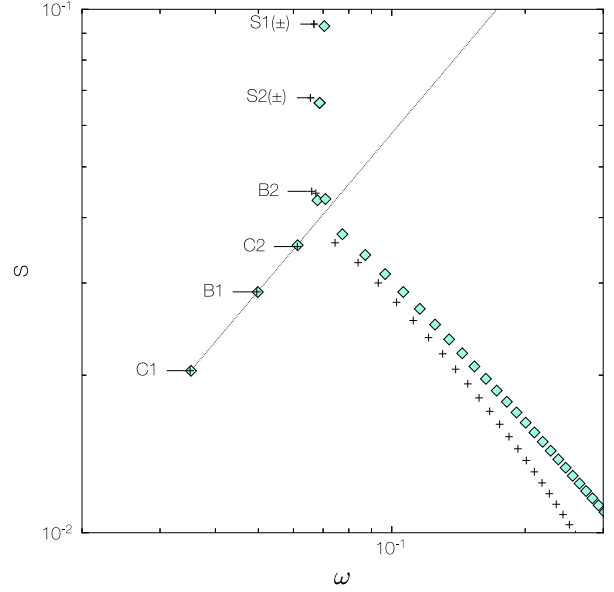
$$Y_n(Z) = \sum_{m=0}^n B_m Z^m, \quad (56)$$

where  $B_m$  are constant coefficients. Substituting this into equation (55), we arrive at the recurrence relation between  $B_m$  and  $B_{m+2}$ ,

$$B_{m+2} = \frac{(a_1 - 2\xi_n)m - (a_2 + \xi_n)}{(m+1)(m+2)} B_m, \quad (57)$$

where we have relabelled  $\xi \rightarrow \xi_n$ . For physical solutions to equation (55), we demand the vertical kinetic energy density of perturbations to remain bound, i.e.  $\rho |v'_Z|^2$  approaches zero for large  $|Z|$ . Then  $Y_n(Z)$  should be a polynomial, as assumed. Thus  $B_{n+2} = 0$  when  $m = n$ , or

$$(a_1 - 2\xi_n)n = a_2 + \xi_n. \quad (58)$$



**Figure 1.** Comparison of growth rates  $s$  and oscillation frequencies  $\omega$  of N13 and this work ( $\beta_\phi \rightarrow \infty$ ). Line denotes analytical solutions obtained in Section 4.2. Diamonds denote numerical solutions obtained in N13. Crosses denote numerical solutions obtained in Section 4.3. Labels B, C, and S represent breathing, corrugation, and surface modes, respectively. Numbers represent fundamental and first overtone modes. Modes reside in the lower right are high-order body modes.

This relation can be also quickly obtained by recognizing equation (55) as the Hermite differential equation (Barker & Latter 2015). We choose the negative root in equation (54) for physical solutions, giving

$$\xi_n = \frac{1}{2}a_1 - \frac{1}{2}\sqrt{a_1^2 + 4a_3}. \quad (59)$$

For a given set of disc parameters ( $q$ ,  $\beta$ ,  $h$ ), equations (51), (58), and (59) can be readily solved to obtain the dispersion relation for the complex frequency  $\sigma_n = \sigma_n(K; n)$  of the  $n$ th mode. The corresponding eigenfunction  $v'_Z$  is then given by equations (52), (56), and (57). We note that the vertical shear rate increases without bound with height in thin discs (equation 48), leading to unbound growth rates (as seen in Fig. 1 discussed below), which violates the low frequency approximation.

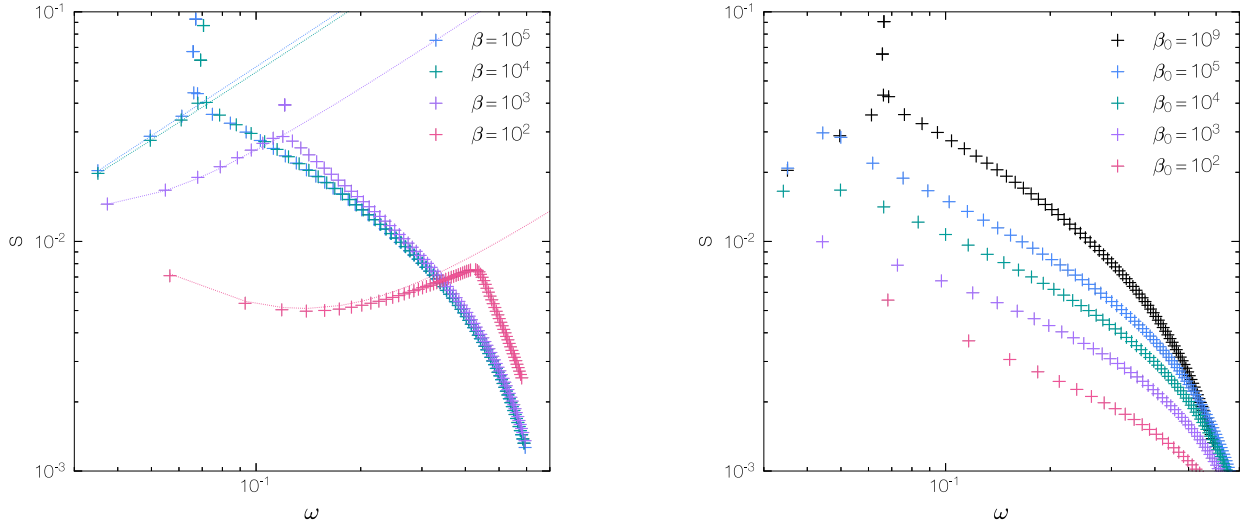
### 4.3 Numerical solutions

We also solve the full linearized equations (42)–(46) numerically. The equations can be written in a form of standard matrix eigenvalue problem,

$$\mathcal{L} \cdot \mathbf{X} + \sigma \cdot \mathbf{X} = 0, \quad (60)$$

where  $\sigma$  is the eigenvalue,  $\mathcal{L}$  is a  $5 \times 5$  matrix of linear operators, and  $\mathbf{X} = [v'_R, v'_\phi, v'_Z, \rho', \Pi']^T$  is a vector of eigenfunctions. We use DEDALUS<sup>2</sup> (Burns et al. 2020), a general purpose spectral code for differential equations, to solve the linear eigenvalue problem. We employ a Chebyshev collocation grid of  $N = 150$  points. Spurious solutions are filtered out and numerical convergence is verified

<sup>2</sup><https://dedalus-project.org/>



**Figure 2.** Growth rates  $s$  and oscillation frequencies  $\omega$  of unstable modes at discrete plasma  $\beta_\phi$  or  $\beta_{\phi 0}$ . Left: a constant- $\beta$  disc. Right: a constant- $B_\phi$  disc. Curves denote analytic solutions (Section 4.2) and crosses denote numerical solutions (Section 4.3).

against double-resolution calculations with  $N = 300$  by using the EIGENTOOLS package.<sup>3</sup>

Unlike the analytic model in Section 4.2, where the disc surface extends to infinity due to the vertically isothermal background state, for numerical solutions we consider a finite vertical domain with  $Z \in [-5, 5]H$ . The boundary conditions imposed at the disc surfaces are

$$v'_Z = 0, \quad \frac{\partial \Pi'}{\partial Z} = 0. \quad (61)$$

The former condition is adopted in the limit of ideal MHD, and the latter being additional conditions when resistivity is included. We have experimented with boundary conditions and found that our main findings are insensitive to them.

## 5 RESULTS

In this section, we present example solutions in the hydrodynamic limit (Section 5.1), the ideal MHD limit (Section 5.2), and the non-ideal MHD limit (Section 5.3). The fiducial parameter values are  $h = 0.05$ ,  $q_T = 1$ ,  $q_D = 1.5$ , and  $K = 35$ . Note that  $q_T = 1$  gives a constant disc aspect ratio  $h$ . We present the non-dimensionalized perturbed quantities as in equation (49). Plasma beta parameters associated with azimuthal and vertical fields are denoted by  $\beta_\phi = 2\mu_0 P/B_\phi^2$  and  $\beta_Z = 2\mu_0 P/B_Z^2$ . We denote mid-plane plasma beta parameters by  $\beta_{\phi 0}$  and  $\beta_{Z0}$ , and mid-plane Elsässer number  $\Lambda_0$ . For reference, values of  $\beta_Z \sim 10^4$  and  $\beta_\phi \sim 10^2$  are found to account for the accretion rate in PPDs (Simon et al. 2013a; Bai 2015). Note, however, as in the discussion above in this section we only consider azimuthal fields, so that  $\beta = \beta_\phi$ . Poloidal fields will be explored in Section 6.

We follow N13 to denote breathing, corrugation, and surface modes as B, C, and S, respectively, with numbers 1 and 2 representing the fundamental and first overtone modes. For clarity, analytic solutions that have discrete modes are plotted as continuous curves.

### 5.1 The hydrodynamic limit

We first compare our numerical and analytical solutions with N13, who considered purely hydrodynamic discs. To this end, we compute the numerical solutions to equations (42)–(46) and analytical solutions given via equation (58) in the hydrodynamic limit ( $\beta_\phi \rightarrow \infty$ ), and compare with numerical solutions to equation (39) in N13. Note that N13 employed the anelastic approximation ( $\partial \rho / \partial t = 0$ ), while we account for full compressibility in numerical solutions. The results are shown in Fig. 1.

For the fundamental and first overtone breathing and corrugation modes (B1, B2, C1, and C2), all three methods yield consistent results. For surface modes and higher order body modes in the lower right of Fig. 1, the two numerical solutions also show consistency, especially at low oscillation frequencies. The analytic solutions for these higher order body modes do not match with numerical solutions due to the lack of a disc surface in the former (Barker & Latter 2015). In practice, the growth rate is limited by the maximum vertical shear rate within the domain (Lin & Youdin 2015), as reflected in the numerical solutions. Overall, the comparison is satisfactory.

### 5.2 The ideal MHD limit

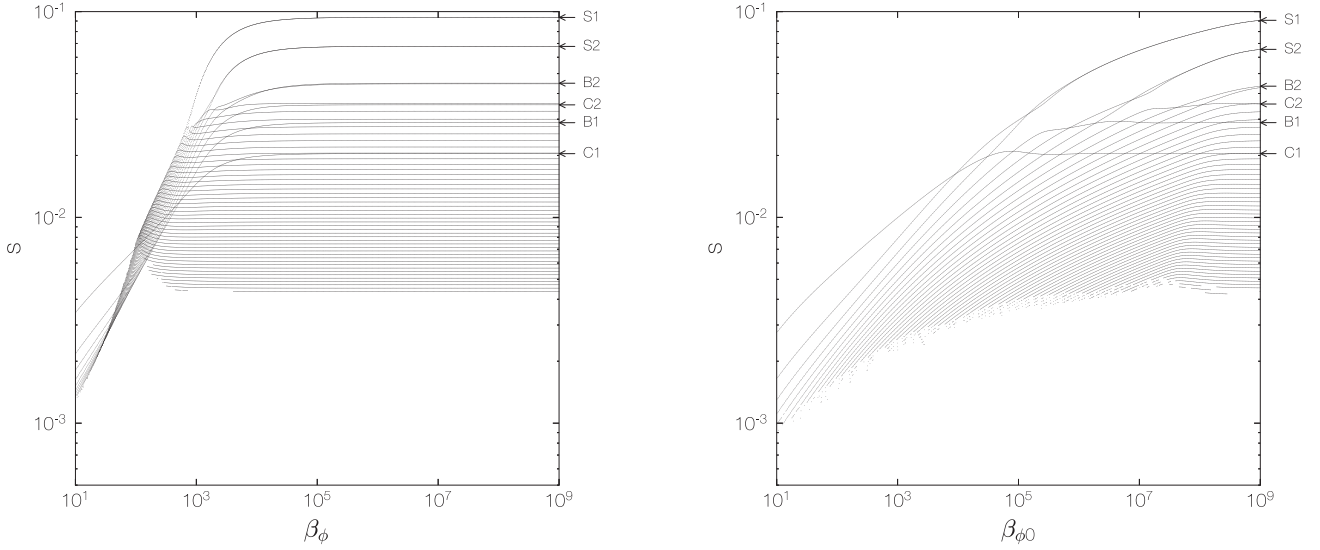
In the ideal MHD limit, we examine the behaviour of the VSI modes as functions of disc magnetizations  $\beta_\phi$  (Section 5.2.1), radial wavenumbers  $K$  (Section 5.2.2), and disc aspect ratios  $h$  (Section 5.2.3).

#### 5.2.1 Disc magnetization

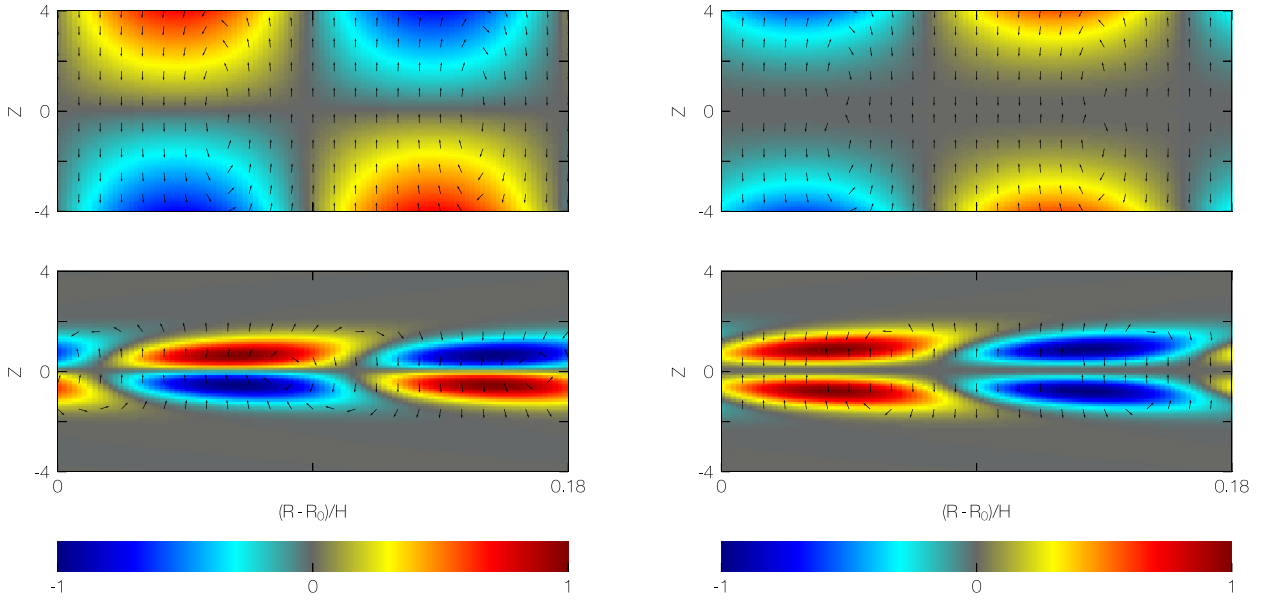
We now examine the strengths of toroidal magnetic fields on the VSI. In Fig. 2, we show example growth rates and frequencies for constant- $\beta$  discs in the left-hand panel and constant- $B_\phi$  discs in the right-hand panel. Similarly, Fig. 3 shows how growth rates of various modes vary with plasma beta. The curves without labels are high-order body modes.

We highlight three major findings. First, strong magnetization reduces the VSI growth. Physically, this is because the gas and the

<sup>3</sup><https://github.com/DedalusProject/eigentools>



**Figure 3.** Growth rates  $s$  of all unstable modes as a function of plasma  $\beta_\phi$  or  $\beta_{\phi 0}$ . Left: a constant- $\beta$  disc. Right: a constant- $B_\phi$  disc.

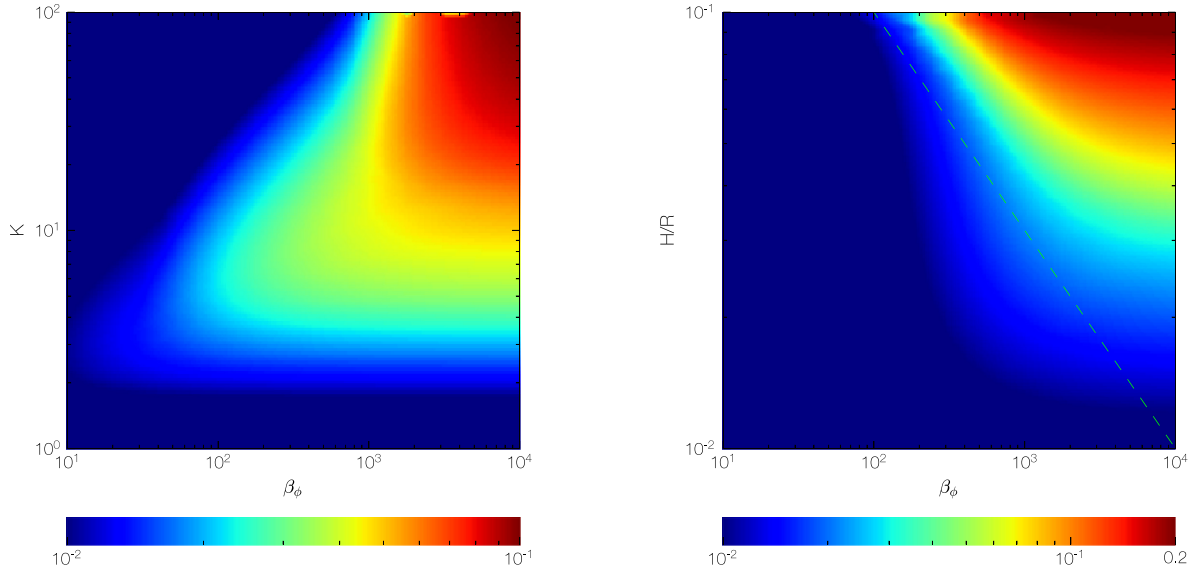


**Figure 4.** Fundamental corrugation modes C1 (left) and fundamental breathing modes B1 (right) at  $\beta_\phi = 10^5$  (top) and  $\beta_\phi = 10^2$  (bottom) in constant- $\beta$  discs. Contours show the magnetic field perturbations,  $\text{Re}\{B'_\phi \exp[iK(R - R_0)]\}/B_\phi$ , normalized by its maximum. Arrows denote perturbed velocity vectors ( $v'_R$ ,  $v'_Z$ ). The radial interval of  $K(R - R_0) = 2\pi$  corresponds to  $0.18H$ .

magnetic fields are perfectly coupled in the limit of ideal MHD, so that magnetic fields impede the free movement of the perturbed gas. Furthermore, surface modes are the first to vanish with strong magnetization. This can be understood by the fact that the stabilizing vertical buoyancy scales as  $Z^2$  for both models as seen in equations (37) and (38), hence the gas is subject to stronger stabilization at the disc surface. Finally, the critical  $\beta_\phi$  to recover hydrodynamic results for a constant- $\beta$  disc,  $\beta_\phi \gtrsim 10^5$ , is smaller than that for the mid-plane value in a constant- $B_\phi$  disc,  $\beta_{\phi 0} \gtrsim 10^9$ . This is because in a constant- $B_\phi$  disc, equation (29) shows that  $\beta_\phi$  decreases with height, so the vertically averaged  $\beta$  is smaller than its mid-plane value.

Fig. 4 shows the flow structure in a constant- $\beta$  disc. The radial domain of  $K(R - R_0) = 2\pi$  corresponds to an interval of  $0.18H$ . The left-hand panels show the fundamental corrugation modes in discs with  $\beta_\phi = 10^5$  (top) and  $\beta_\phi = 10^2$  (bottom). The perturbed vertical velocities show even symmetry about the mid-plane. The right-hand panels are corresponding fundamental breathing modes, where the perturbed vertical velocities have odd symmetry. The contours show the magnetic field perturbations  $\text{Re}\{B'_\phi \exp[iK(R - R_0)]\}/B_\phi$  and is normalized by its maximum value. The perturbed magnetic fields possess opposite symmetry to perturbed vertical velocities. Importantly, we find that strong magnetization confines VSI activity towards the mid-plane since the stabilizing vertical buoyancy in-





**Figure 5.** Contours of maximum growth rates in logarithmic scale as functions of  $\beta_\phi$  and radial wavenumber  $K$  (left) or disc aspect ratio  $h$  (right) in a constant- $\beta$  disc. Dashed line represents  $\beta_\phi = h^{-2}$ .

creases with height. The same arguments also apply to a constant- $B_\phi$  disc.

### 5.2.2 Radial wavenumber

The left-hand panel of Fig. 5 depicts contours of maximum growth rates as a function of  $\beta_\phi$  and radial wavenumber  $K$  for constant- $\beta$  discs. A critical  $\beta_c \sim 10^3$  can be defined to separate the discs into two regimes. In constant- $B_\phi$  discs it is  $\beta_c \sim 10^5$ . For  $\beta_\phi \gtrsim \beta_c$ , the maximum growth rate is a monotonically increasing function of  $K$ , whereas for  $\beta_\phi \lesssim \beta_c$ , the maximum growth rate peaks at some intermediate  $K$ . We explain below that  $\beta_c$  is in fact the critical disc magnetization below which surface modes are quenched. In Fig. 3, we see that at  $\beta_\phi \gtrsim \beta_c$ , surface modes, which prefer very small radial wavelengths, dominate the maximum growth rates resulting in fast growth rates at large  $K$ . At  $\beta_\phi \lesssim \beta_c$ , surface modes are suppressed, while body modes that prefer longer radial wavelengths persist, and thus maximum growth rates appear at intermediate radial wavenumbers.

### 5.2.3 Disc aspect ratio

In the right-hand panel of Fig. 5, we show contours of maximum growth rates as functions of  $\beta_\phi$  and disc aspect ratio  $h$ , again for constant- $\beta$  discs. The maximum growth rates increase with the disc aspect ratio for a given  $\beta_\phi$ . Requiring modes to fit into the vertical height of the disc, a lower limit can be placed on  $\beta_\phi$  for the VSI to operate,  $\beta_{\min} \gtrsim (-R\partial \ln \Omega / \partial Z)^{-2}$  (LP18). This is set by the vertical shear rate, and can be simplified to  $\beta_{\min} \gtrsim h^{-2}$  using equation (48). Note, however, this criterion was derived for purely poloidal background fields in a local approximation, while we consider purely toroidal magnetic fields in a vertically global disc. Nevertheless, we find the local condition  $\beta_\phi = h^{-2}$ , shown in Fig. 5 as the dashed line, successfully predicts the quenching of the VSI in our disc model.

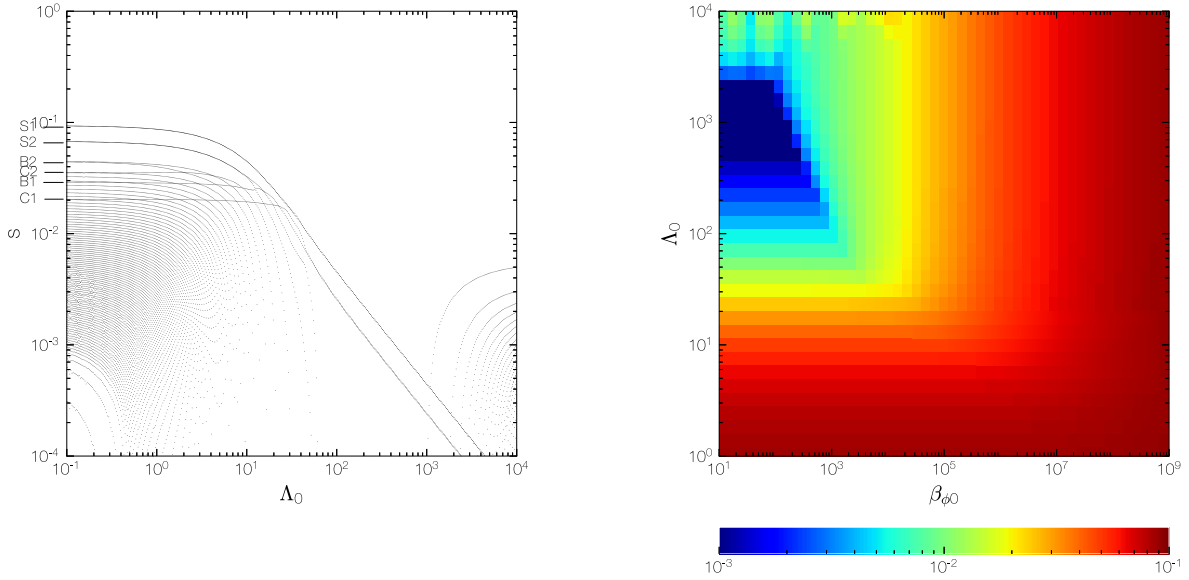
### 5.3 The non-ideal MHD limit

In this section, we show that the VSI can be revived when non-ideal MHD effects are included. To assure the existence of equilibrium solutions, a constant- $B_\phi$  disc model is employed (Section 2.3.2). With purely toroidal magnetic fields in axisymmetric discs, the three non-ideal MHD effects reduce to only Ohmic resistivity, because the Hall effect vanishes, and ambipolar diffusion acts as an effective resistivity with field dependency (Section 2.1.1). Therefore, we only explore the dependency of Ohmic Elsässer number  $\Lambda$ , while we expect the same results apply to ambipolar diffusion. We take the diffusivity  $\eta$  to be constant so that the Elsässer number increases with  $|Z|$ , as shown in equation (11).

In the left-hand panel of Fig. 6, we show the growth rates of all unstable modes as a function  $\Lambda_0$  at  $\beta_{\phi 0} = 10^2$ . The labels correspond to surface and body modes in the hydrodynamic limit (Fig. 1), which is recovered for small  $\Lambda_0$  or strong Ohmic resistivity. On the other hand,  $\Lambda_0 \rightarrow \infty$  tends to the ideal MHD limit. We find for  $\Lambda_0 \gtrsim 10^3$ , the surface modes vanish and the growth rate of body modes is significantly reduced due to strong magnetization. As  $\Lambda_0$  declines from larger values, the growth rates of these body modes drop at around  $\Lambda_0 \sim 10^3$ , then they re-emerge and converge to hydrodynamic results. The growth rates of all modes converge to hydrodynamic results for  $\Lambda_0 \lesssim 10$ , with the transition starting at  $\Lambda_0 \sim 10^2$ . Local analyses demonstrate that the stabilizing effect by magnetic fields will be overcome by magnetic diffusion when  $\Lambda \lesssim h^{-1}$  ( $=20$  in our fiducial disc) for a mode with growth rate  $\sim h\Omega$  (LP18).<sup>4</sup> This is in agreement with our results, though the growth rates from our solutions are only reduced rather than completely suppressed.

The right-hand panel of Fig. 6 shows the maximum growth rates as functions of  $\beta_{\phi 0}$  and  $\Lambda_0$ . For  $\beta_{\phi 0} > 10^3$ , the maximum growth rate is a monotonically decreasing function with increasing  $\Lambda_0$ , whereas for  $\beta_{\phi 0} < 10^3$ , the maximum growth rate has its minimum resides at some intermediate  $\Lambda_0$ , corresponding to the left-hand panel of

<sup>4</sup>Equation (64) of LP18 contains a typographical error, the corrected expression is  $E_\eta \lesssim 1/q$  (Latter, private communication).



**Figure 6.** The effect of Ohmic resistivity on the VSI growth rate. Left: growth rate  $s$  versus Ohmic Elsässer number  $\Lambda_0$  at  $\beta_{\phi 0} = 10^2$ . Right: contour of maximum growth rates in logarithmic scales as functions of plasma  $\beta_{\phi 0}$  and Ohmic Elsässer number  $\Lambda_0$ .

Fig. 6. The hydrodynamic result is recovered for sufficiently weak fields ( $\beta_{\phi 0} \gtrsim 10^5$ ) or sufficiently strong resistivity ( $\lambda_0 \lesssim 10$ ).

## 6 PURELY POLOIDAL BACKGROUND MAGNETIC FIELDS

The above analyses focus on discs threaded by a toroidal magnetic field, which is expected to dominate over poloidal field strengths in PPDs (e.g. Bai 2017; Béthune et al. 2017; Cui & Bai 2020). However, the presence of a poloidal field, even weak, can lead to new effects such as MHD disc winds and the MRI (Bai 2013; Simon et al. 2013b; Gressel et al. 2020). Specifically, the surface layers in outer regions of PPDs are likely sufficiently ionized by stellar far-ultraviolet (FUV) radiation to trigger the MRI (Perez-Becker & Chiang 2011; Simon et al. 2013a,b; Bai 2015). These regions are also prone to the VSI since the vertical shear rate increases with height. In this section, we investigate the VSI modes in a disc with purely poloidal magnetic fields,  $\mathbf{B} = (B_R, 0, B_Z)$ . In Section 6.1, we study the effects of poloidal magnetic fields on the VSI in a vertically global disc model. In Section 6.2, we compare the MRI with the VSI in a local disc model. Although toroidal fields are absent in the background, it is allowed in the perturbed state.

### 6.1 Vertically global model

We make several simplifying assumptions to establish disc equilibria with a purely poloidal magnetic field.

(i) The Lorentz force in the momentum equation is ignored because the disc is weakly magnetized. We therefore use equilibrium solutions for  $\Omega$  and  $\rho$  in the hydrodynamic limit (equations 26 and 27 with  $\beta \rightarrow \infty$ , see also N13).

(ii) We assume thin discs and consider  $h \ll 1$ .

(iii) A constant background vertical magnetic field  $B_Z$  is assumed, and we seek the required equilibrium radial magnetic field  $B_R$ , as follows.

The equilibrium magnetic fields must satisfy solenoidal condition and induction equation. Considering only Ohmic resistivity with a constant diffusivity, the equilibrium induction equation is

$$0 = (\mathbf{B} \cdot \nabla) \mathbf{v}_\phi - (\mathbf{v}_\phi \cdot \nabla) \mathbf{B} + \eta \nabla^2 \mathbf{B}, \quad (62)$$

where  $\mathbf{v}_\phi = R\Omega\hat{\phi}$ . In the thin disc approximation, the gradients of  $v_\phi$  are

$$\frac{\partial v_\phi}{\partial Z} \simeq -\frac{1}{2} \Omega_K q_T \frac{Z}{R}, \quad (63)$$

$$\frac{\partial v_\phi}{\partial R} \simeq -\frac{1}{2} \Omega_K. \quad (64)$$

The induction equation can be satisfied by the radial field,

$$B_R = -\frac{1}{3} q_T \frac{Z}{R} B_Z. \quad (65)$$

Notice that this field configuration is not subject to Ohmic diffusion and satisfies the solenoidal condition,

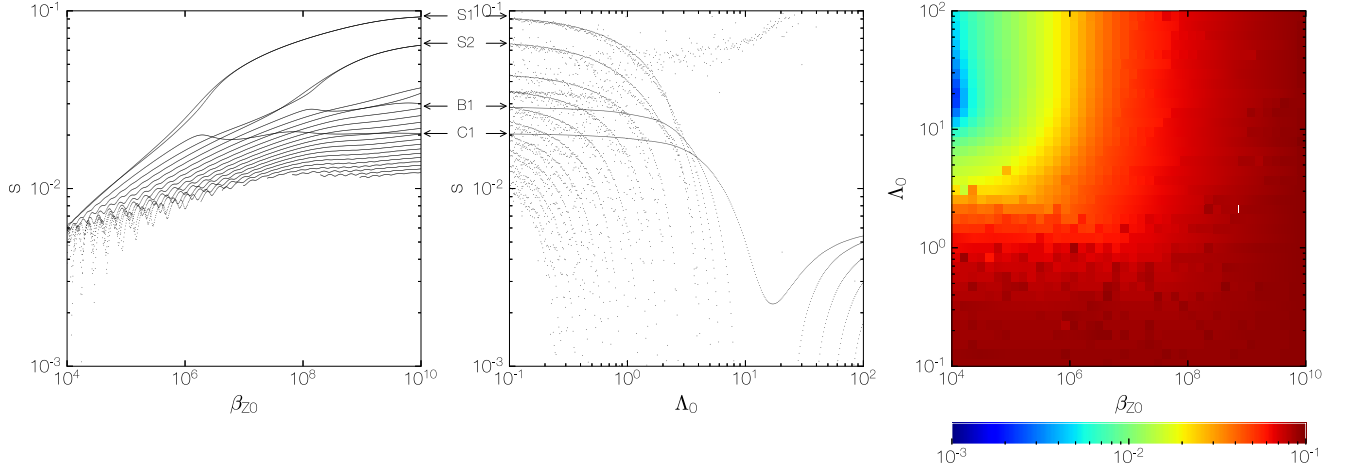
$$\frac{1}{R} \frac{\partial(RB_R)}{\partial R} + \frac{\partial B_Z}{\partial Z} = 0. \quad (66)$$

Therefore, an approximate equilibrium magnetic field configuration is obtained. The equilibrium solution, equation (65), resembles equation (50) in LP18. Since  $|B_R| \sim O(h)|B_Z|$ , the strength of the magnetic field is dominated by the vertical field, which is assumed a constant so that this is similar to the constant- $B_\phi$  discs considered in Section 2.3.

With a poloidal field it is not possible to map the problem to adiabatic hydrodynamics. We therefore work with the MHD equations directly. The radial derivatives of background quantities are omitted. The set of linearized equations is

$$\sigma \frac{\rho'}{\rho} + ik_R v'_R + \frac{dv'_Z}{dZ} + \frac{d \ln \rho}{dZ} v'_Z = 0, \quad (67)$$

$$\sigma v'_R - 2\Omega v'_\phi + ik_R \frac{P'}{\rho} + \frac{ik_R}{\mu_0 \rho} \mathbf{B} \cdot \mathbf{B}' - \frac{1}{\mu_0 \rho} \left[ ik_R B_R B'_R + B_Z \frac{dB'_R}{dZ} + \frac{dB_R}{dZ} B'_Z \right] = 0, \quad (68)$$



**Figure 7.** Left: growth rates  $s$  of all unstable modes as a function of mid-plane  $\beta_{Z0}$  in the ideal MHD limit. Middle: growth rates  $s$  versus mid-plane Ohmic Elsässer number  $\Lambda_0$  at  $\beta_{Z0} = 10^4$ . Right: contour of maximum growth rates as functions of  $\beta_{Z0}$  and Ohmic Elsässer number  $\Lambda_0$ .

$$\sigma v'_\phi + \frac{\kappa^2}{2\Omega} v'_r + \frac{dv_\phi}{dZ} v'_z - \frac{1}{\mu_0 \rho} \left[ ik_R B_R B'_\phi + B_Z \frac{dB'_\phi}{dZ} \right] = 0, \quad (69)$$

$$\sigma v'_z + \frac{1}{\rho} \frac{dP'}{dZ} - \frac{\rho'}{\rho^2} \frac{dP}{dZ} + \frac{1}{\mu_0 \rho} \frac{\partial \mathbf{B} \cdot \mathbf{B}'}{\partial Z} - \frac{1}{\mu_0 \rho} \left[ ik_R B_R B'_Z + B_Z \frac{dB'_Z}{dZ} \right] = 0, \quad (70)$$

$$\sigma B'_R - \left[ ik_R B_R + B_Z \frac{d}{dZ} \right] v'_r + \left[ ik_R v'_r + \frac{dv'_z}{dZ} + v'_z \frac{\partial}{\partial Z} \right] B_R + \left[ k_R^2 - \frac{d^2}{dZ^2} \right] \eta B'_R = 0, \quad (71)$$

$$\sigma B'_\phi - \left[ ik_R B_R + B_Z \frac{d}{dZ} \right] v'_\phi - \left[ \frac{\kappa^2}{2\Omega} - 2\Omega \right] B'_R - \frac{\partial v_\phi}{\partial Z} B'_Z + \left[ k_R^2 - \frac{d^2}{dZ^2} \right] \eta B'_\phi = 0, \quad (72)$$

$$\sigma B'_Z - \left[ ik_R B_R + B_Z \frac{d}{dZ} \right] v'_z + \left[ ik_R v'_r + \frac{dv'_z}{dZ} \right] B_Z + \left[ k_R^2 - \frac{d^2}{dZ^2} \right] \eta B'_Z = 0, \quad (73)$$

$$P' = \rho' c_s^2. \quad (74)$$

Note that there is now a magnetic tension force in the momentum equations.

We solve the linear eigenvalue problem, in its dimensionless form, numerically using the spectral method described in Section 4.3. A resolution of  $N = 100$  is used. Spurious solutions are filtered out with double resolution calculations. Boundary conditions imposed at upper and lower disc surfaces for ideal MHD limit are (Gammie & Balbus 1994; Sano & Miyama 1999)

$$\rho' = 0, \quad B'_R = 0, \quad B'_\phi = 0. \quad (75)$$

Additional conditions are imposed with Ohmic resistivity,

$$P' = 0, \quad B'_Z = 0. \quad (76)$$

The quantities  $\sigma$ ,  $Z$ , and  $K$  reported below are non-dimensionalized as in equation (49). With the inclusion of vertical magnetic fields,

MRI modes emerge in the numerical solutions. Unlike the VSI, however, MRI modes are not overstable even with resistivity, i.e.  $\text{Im}(\sigma) \equiv \omega = 0$  (Sano & Miyama 1999), while it can be seen in Fig. 2 that VSI modes generally have  $0.01 < \omega < 1$ , hence the MRI modes do not contaminate the numerical results of the VSI.

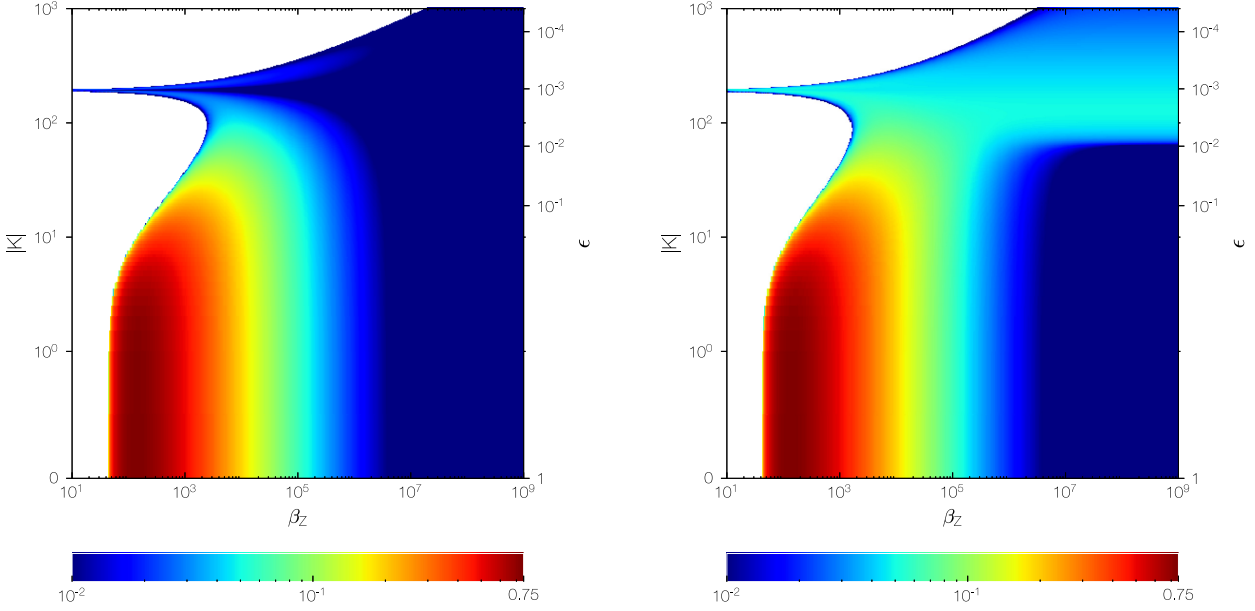
The left-hand panel of Fig. 7 shows growth rates of all unstable modes as a function of disc magnetization  $\beta_{Z0}$  in the ideal MHD limit. Consistent with purely the toroidal field model, strong magnetization suppresses VSI growth and surface modes are the first to vanish with increasing field strengths. The critical  $\beta_{Z0}$  to recover hydrodynamic results is even larger,  $\beta_{Z0} \gtrsim 10^{10}$ , because the total magnetic field strength increases over height as radial magnetic field develops away from the mid-plane in equation (65).

Growth rates including Ohmic resistivity are shown in the middle and right-hand panels of Fig. 7. The middle panel shows the growth rates of all unstable modes as a function of Ohmic Elsässer number  $\Lambda_0$  at  $\beta_{Z0} = 10^4$ . Nearly all the VSI modes are suppressed when Ohmic resistivity is weak at  $\Lambda_0 \gtrsim 10$ . The VSI modes start to grow when  $\Lambda_0 \lesssim 10$ . Fundamental body modes B1 and C1 converge to the hydrodynamic growth rates at  $\Lambda_0 \sim 1$ . On the other hand, surface modes and high-order body modes show slower transitions to their hydrodynamic growth rates, requiring  $\Lambda_0 \sim 0.1$ . The right-hand panel of Fig. 7 shows the maximum growth rate as a function of  $\beta_{Z0}$  and  $\Lambda_0$ . Large  $\Lambda_0 \sim 10^2$  corresponds to the ideal MHD limit, where maximum growth rate drops to  $\sim 5 \times 10^{-3}$  for  $\beta_{Z0} = 10^4$ . It can be seen that  $\Lambda_0 < 1$  is required for the fastest growing modes to recover hydrodynamic results in a wide range of  $\beta_{Z0}$  from  $10^4$  to  $10^{10}$ .

The above results are similar to that for toroidal fields, which indicates that the qualitative effect of a magnetic field and resistivity on the VSI does not depend on the background field geometry.

## 6.2 VSI versus MRI

To better compare the VSI and the MRI, we perform a vertically local linear analysis in the incompressible limit with Ohmic resistivity. In this local model, background quantities are assumed to be uniform and its values set to that in the vertically global model (Section 6.1) at some fiducial height. We take the vertical shear rate  $R\partial\Omega^2/\partial Z$  as an input parameter. We consider axisymmetric perturbations that are proportional to  $\exp(\sigma t + ik_R R + ik_Z Z)$ . The wavenumber vector is denoted by  $\mathbf{k} = (k_R, 0, k_Z)$ .



**Figure 8.** Contours of growth rates as functions of  $\beta_Z$  and radial wavenumber  $|K|$  by equation (84) in the ideal MHD limit. The vertical wavenumber is set to be  $k_Z = 2\pi/H$ , and  $\epsilon = k_Z^2/k^2$ . Left: no vertical shear  $R\partial\Omega^2/\partial Z = 0$ . Right: with vertical shear  $R\partial\Omega^2/\partial Z = -0.1$ .

In the incompressible limit, the linearized equations derived from equations (1), (2), and (7) become

$$ik_R v'_R + ik_Z v'_Z = 0, \quad (77)$$

$$\sigma v'_R - 2\Omega v'_\phi + ik_R \frac{P'}{\rho} + \frac{ik_R}{\mu_0 \rho} \mathbf{B} \cdot \mathbf{B}' - \frac{iB'_R}{\mu_0 \rho} \mathbf{k} \cdot \mathbf{B} = 0, \quad (78)$$

$$\sigma v'_\phi + \frac{\kappa^2}{2\Omega} v'_R + \frac{dv_\phi}{dZ} v'_Z - \frac{iB'_\phi}{\mu_0 \rho} \mathbf{k} \cdot \mathbf{B} = 0, \quad (79)$$

$$\sigma v'_Z + ik_Z \frac{P'}{\rho} + \frac{ik_Z}{\mu_0 \rho} \mathbf{B} \cdot \mathbf{B}' - \frac{iB'_Z}{\mu_0 \rho} \mathbf{k} \cdot \mathbf{B} = 0, \quad (80)$$

$$\tilde{\sigma} B'_R - i\mathbf{k} \cdot \mathbf{B} v'_R = 0, \quad (81)$$

$$\tilde{\sigma} B'_\phi - i\mathbf{k} \cdot \mathbf{B} v'_\phi - \left[ \frac{\kappa^2}{2\Omega} - 2\Omega \right] B'_R - \frac{dv_\phi}{dZ} B'_Z = 0, \quad (82)$$

$$\tilde{\sigma} B'_Z - i\mathbf{k} \cdot \mathbf{B} v'_Z = 0, \quad (83)$$

where

$$k^2 = k_R^2 + k_Z^2,$$

$$\tilde{\sigma} = \sigma + \eta k^2.$$

The above equations give a dispersion relation

$$\begin{aligned} & [\sigma \tilde{\sigma} + (\mathbf{k} \cdot \mathbf{v}_A)^2]^2 + \frac{k_Z^2}{k^2} (\kappa^2 - A) [\tilde{\sigma}^2 + (\mathbf{k} \cdot \mathbf{v}_A)^2] \\ & - 4\Omega^2 (\mathbf{k} \cdot \mathbf{v}_A)^2 \frac{k_Z^2}{k^2} = 0, \end{aligned} \quad (84)$$

where

$$A = \frac{k_R}{k_Z} \frac{R\partial\Omega^2}{\partial Z}.$$

Equation (84) generalizes that of LP18 to include Ohmic resistivity. Appendix A explores this dispersion relation in more detail in the

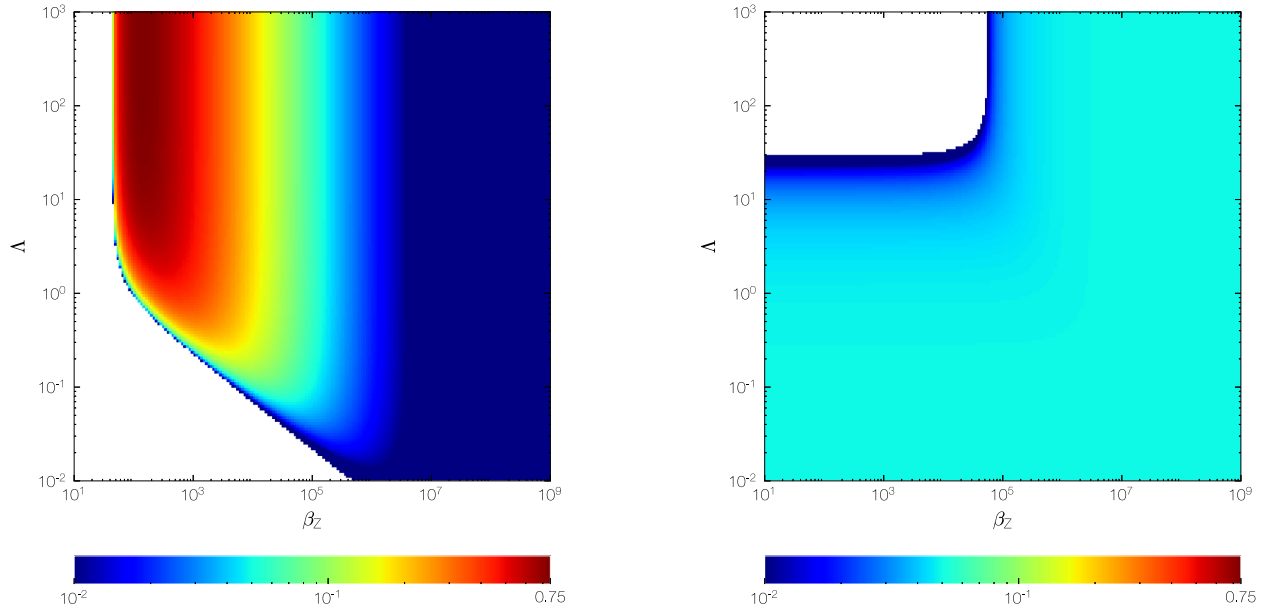
limit  $|k_R B_R| \ll |k_Z B_Z|$ . Here, we solve equation (84) in full to investigate the dominance of MRI and VSI as functions of disc magnetizations  $\beta_Z$ , radial wavenumbers  $K$ , and Elsässer numbers  $\Lambda$ . A vertical wavenumber  $k_Z = 2\pi/H$  is fixed in numerical calculations. VSI modes require  $k_R$  and  $k_Z$  with opposite signs if  $\partial\Omega/\partial Z < 0$  (N13; LP18), as we will consider below, hence we use  $|K|$  to denote the absolute value of  $K$ . Other background quantities are evaluated at  $Z = 2H$ .

The left-hand panel of Fig. 8 shows local growth rates of MRI modes in the ideal MHD limit ( $\eta = 0$ ) without vertical shear ( $R\partial\Omega^2/\partial Z = 0$ ). In the white regions, the MRI is quenched by strong magnetizations. It can be seen that the MRI modes prefer small  $|K|$ , with a maximum growth rate of  $s = 0.75$  at  $\beta_Z = 137$ . In the right-hand panel of Fig. 8, we show growth rates of VSI modes by setting  $R\partial\Omega^2/\partial Z = -0.1$ . The fastest growing VSI modes prefer weak magnetization and large  $|K|$ , in contrast to fast growing MRI modes. Local VSI modes with  $K = 150$  and  $\beta_Z = 10^9$  have a growth rate  $s = 0.05$ , which is less than that of the fastest growing surface mode  $s = 0.094$  obtained from vertically global analysis shown in the right-hand panel of Fig. 3.

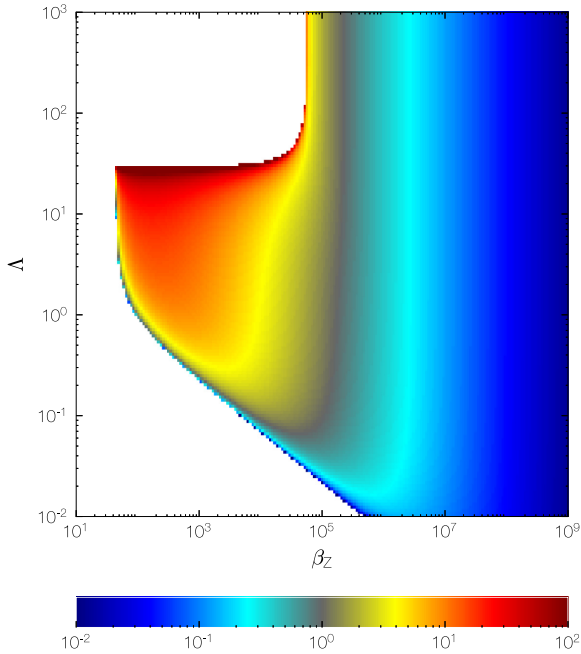
We next consider resistive discs. In the left-hand panel of Fig. 9, we show growth rates as functions of  $\beta_Z$  and  $\Lambda$  for MRI modes by setting  $K = 0$ . In contrast to the VSI, MRI growth rates decline towards small  $\Lambda$  as the MRI is dampened. In the right-hand panel of Fig. 9, we show growth rates of mostly VSI modes by setting  $K = 150$ . For  $\beta_Z \gtrsim 10^5$ , we obtain VSI growth rates in the hydrodynamic limit. For  $\beta_Z \lesssim 10^5$ , the VSI is dampened for  $\Lambda \gtrsim 10$ , but revived with small  $\Lambda$  or strong resistivity.

Finally, in Fig. 10, we show the ratio of MRI growth rates obtained from the left-hand panel of Fig. 9 to VSI growth rates obtained from the right-hand panel of Fig. 9. For  $\beta_Z \gtrsim 10^5$  the VSI dominates over the MRI. For  $\beta_Z \lesssim 10^5$ , which includes PPDs with typical  $\beta_Z \sim 10^4$ , the VSI dominates if  $\Lambda \lesssim 0.09$ .





**Figure 9.** Contours of growth rates as functions of  $\beta$  and Ohmic Elsasser number  $\Lambda$  by equation (84). The vertical wavenumber and vertical shear are set to be  $k_z = 2\pi/H$  and  $R\partial\Omega^2/\partial Z = -0.1$ . Left: MRI modes ( $|K| = 0$ ). Right: VSI modes ( $|K| = 150$ ).



**Figure 10.** The ratio of MRI growth rates to VSI growth rates computed from local dispersion relation equation (84).

## 7 DISCUSSION

### 7.1 Comparison with previous works

LP18 carried out local linear analyses of the VSI in the ideal MHD limit with an exact background equilibrium solution for a purely poloidal field. In this work, we set up global equilibria for a purely azimuthal field. Comparing our numerical solutions to local analytical results, we find that the overall mode behaviour that

magnetization can stabilize the VSI is in agreement with each other. Global numerical simulations of magnetized PPDs carried out by Cui & Bai (2020) indeed show that magnetism tends to suppress the VSI growth, whereas ambipolar diffusion acts to revive VSI modes. Their simulations also show the absence of surface modes, which is consistent with our findings that surface modes are the first to be dampened with increasing field strengths.

Global simulations of PPDs initially threaded by a large-scale poloidal magnetic field suggest a field configuration dominated by the toroidal component once the disc reaches a quasi-steady state (e.g. Bai 2017; Béthune et al. 2017; Cui & Bai 2020). This suggests that, as far as the VSI is concerned, it is the disc model with an azimuthal field that is more relevant, as employed in most of this paper. Furthermore, the magnetization in our toroidal field model is parametrized by  $\beta$ , which does not depend on the orientation of the magnetic field. Our results are thus applicable to the aforementioned simulations wherein the toroidal field reverses polarity across the disc mid-plane because of the Keplerian shear.

### 7.2 Application to PPDs

In the outer part of the PPDs ( $\gtrsim 30$  au), ambipolar diffusion is the dominant non-ideal MHD effect, with Elsasser numbers approximately unity. For a purely azimuthal field, ambipolar diffusion acts as an effective resistivity with field dependency (see Section 2.1.1). Hence, our results for Ohmic resistivity are also applicable to ambipolar diffusion. A value of  $\Lambda_0 = 1$  and  $\beta_{\phi 0} = 10^2$  gives a maximum growth rate of  $s = 0.087$  (Fig. 6), which is close to the hydrodynamic result,  $s = 0.094\Omega_K$ . In the inner part of the PPDs, Ohmic resistivity becomes the dominant non-ideal MHD effect, though the Hall effect also contributes. At 2 au, the mid-plane  $\Lambda_0 = 5 \times 10^{-4}$  (Bai 2017) gives a maximum growth rate of  $s = 0.094$  for a wide range from  $\beta_{\phi 0} = 10$  to  $10^9$  (Fig. 6), as a small  $\Lambda_0$  enables the recovery of hydrodynamic results.

Our locally isothermal disc models, which correspond to instantaneous cooling, favour the VSI because there is no stabilizing effect

from vertical gas buoyancy (N13; Lin & Youdin 2015). However, in a realistic PPD cooling time-scales are finite and is sensitive to stellar irradiation and dust properties (Malygin et al. 2017; Flock et al. 2020; Pfeil & Klahr 2020). In magnetized discs, the magnetic field will provide extra stabilization via magnetic buoyancy in addition to gas buoyancy. Thus, we expect that in PPDs the required cooling time may be shorter than that estimated based on purely hydrodynamic models (Lin & Youdin 2015), unless  $\Lambda_0$  is small enough to diminish the stabilizing effect from magnetic fields. Detailed analyses should be conducted to give new critical cooling time-scales in magnetized discs.

The analysis we present with Ohmic resistivity points to future directions in including ambipolar diffusion and the Hall effect, the latter of which requires a poloidal field. A few obstacles and complications need to be resolved when incorporating these two non-ideal MHD effects. First, it is difficult to find appropriate background equilibria in a global model because of the vertical shear, especially for in presence of poloidal magnetic fields (Ogilvie 1997). Furthermore, ambipolar diffusion gives rise to anisotropic damping and introduces the ambipolar shear instability (Blaes & Balbus 1994; Kunz & Balbus 2004; Kunz 2008). The Hall effect will further introduce the Hall shear instability, and its effect is polarity dependent (Balbus & Terquem 2001; Kunz 2008). All of these effects will complicate the problem and deserve step-by-step analyses in local and global disc models in the future.

### 7.3 Implications for dust dynamics

Small dust grains tend to settle towards the disc mid-plane (Dubrulle et al. 1995). However, the VSI drives turbulence that can vertically mix up dust particles (Stoll & Kley 2016; Flock et al. 2017, 2020). On the other hand, our results show that strong magnetization can stabilize the VSI away from the disc mid-plane, implying a limit on the vertical extent of the ensuing VSI turbulence and therefore a maximum dust layer thickness,  $H_{d, \max}$ . For definiteness, consider a purely toroidal field with constant  $\beta$  and neglect non-ideal MHD effects. The destabilizing vertical shear,  $R\partial\Omega^2/\partial Z$ , competes against the stabilizing magnetic buoyancy,  $N_z^2$ . We therefore expect the VSI to be suppressed where  $|R\partial\Omega^2/\partial Z|/N_z^2 < \zeta$ , where  $\zeta$  is some critical ratio.<sup>5</sup> Using equations (25) and (37), we estimate

$$\frac{H_{d, \max}}{H} = \frac{\beta h |q_T|}{\zeta}, \quad (85)$$

assuming  $\beta \gg 1$  and a thin disc. The example in Fig. 4 with  $\beta = 10^2$ ,  $h = 0.05$ , and  $q_T = 1$  shows that gas motions are negligible for  $|Z| \gtrsim 2H$ , which suggest  $\zeta \simeq 2.5$ .

One may ask if  $H_{d, \max}$  can be made sufficiently small to constrain particles to a dense mid-plane layer that can undergo, for example, the streaming instability and hence facilitate planetesimal formation (Youdin & Goodman 2005; Johansen, Youdin & Mac Low 2009). For dynamical growth, the streaming instability requires a local dust-to-gas mass ratio  $\nu \gtrsim 1$ . The metallicity is  $\Sigma_d/\Sigma_g = \nu H_d/H$ , where  $\Sigma_d$  and  $\Sigma_g$  are the dust and gas surface densities, respectively, and  $H_d$  is the characteristic dust scale height. Thus, if we take  $H_d = H_{d, \max}/2$ , then

$$\beta = \frac{2\zeta}{\nu} \frac{\Sigma_d}{|q_T| h \Sigma_g}. \quad (86)$$

<sup>5</sup>For the hydrodynamic VSI,  $\zeta \sim \Omega t_{\text{cool}}$ , where  $t_{\text{cool}}$  is the cooling time-scale (Lin & Youdin 2015).

Inserting typical PPD values  $(q_T, h, \Sigma_d/\Sigma_g) \simeq (1, 0.05, 0.01)$ , we find an equipartition field strength,  $\beta = 1$ , would be required to confine dust into a thin layer such that  $\nu \sim 1$  by quenching the VSI elsewhere. Such a strong field is unrealistic for PPDs (e.g. Simon et al. 2013a; Bai 2015). This suggests that magnetic fields do not affect the vertical dust structure in PPDs through its geometric effect on the VSI.

Instead, magnetic effects likely manifest through weakening the VSI and hence the ensuing turbulence, as found in this work and Cui & Bai (2020). This determines  $H_d \simeq \sqrt{\delta_z/\text{St}} H_g$  (Dubrulle et al. 1995), where  $\text{St}$  is the particle Stokes number and  $\delta_z$  is the dimensionless vertical diffusion coefficient associated with VSI turbulence. We expect  $\delta_z$  to drop with stronger magnetization and to increase with stronger non-ideal MHD effects. This relation should be calibrated with future simulations, which can then be used to estimate magnetic field strengths from the vertical distribution of dust in PPDs.

## 8 CONCLUSIONS

In this work, we perform linear analyses of the VSI under the ideal MHD limit and with Ohmic resistivity. A vertically global and radially local disc model is employed to properly accommodate the characteristic VSI modes of elongated vertical wavelengths. A locally isothermal equation of state is assumed to better focus on the effect of magnetism. Our main findings are summarized as follows.

- (i) In the ideal MHD limit, magnetic fields operate as a stabilizing effect to suppress the growth of VSI modes. Surface modes are the first to vanish rather than body modes with increasing magnetic field strengths. Ohmic resistivity acts as destabilizing effect to assist the VSI growth.
- (ii) In weakly magnetized discs, surface modes show maximum growth rates at large radial wavenumbers, while in strongly magnetized discs, surface modes are dampened, while body modes are dominant and prefer intermediate radial wavenumbers. Large disc aspect ratios or vertical shear rates lead to fast VSI growth.
- (iii) The MRI modes appear when a poloidal magnetic field is present. In the local analysis, we find that MRI and VSI modes dominate at different  $\beta_z$  and  $\Lambda$  in the ideal MHD limit. MRI prefers relatively strong disc magnetizations and small radial wavenumbers. The VSI modes are most effective at weak magnetizations and large radial wavenumbers. With Ohmic resistivity, a typical value of  $\beta_z = 10^4$  in PPDs results in a critical  $\Lambda \lesssim 0.09$  for the dominance of the VSI.

## ACKNOWLEDGEMENTS

We are pleased to thank the anonymous referee for helpful comments and Xue-Ning Bai, Henrik Latter, and Gordon Ogilvie for fruitful discussions. CC acknowledges the support from Department of Applied Mathematics and Theoretical Physics at University of Cambridge. M-KL is supported by the Ministry of Science and Technology of Taiwan under grant 107-2112-M-001-043-MY3 and an Academia Sinica Career Development Award (AS-CDA-110-M06). Software used: DEDALUS (Burns et al. 2020) and EIGENTOOLS package.

## DATA AVAILABILITY

The data underlying this paper will be shared on reasonable request to the corresponding author.

## REFERENCES

- Arlt R., Urpin V., 2004, *A&A*, 426, 755
- Armitage P. J., 2011, *ARA&A*, 49, 195
- Bai X.-N., 2011, *ApJ*, 739, 51
- Bai X.-N., 2013, *ApJ*, 772, 96
- Bai X.-N., 2015, *ApJ*, 798, 84
- Bai X.-N., 2017, *ApJ*, 845, 75
- Bai X.-N., Stone J. M., 2011, *ApJ*, 736, 144
- Bai X.-N., Stone J. M., 2013, *ApJ*, 769, 76
- Bai X.-N., Ye J., Goodman J., Yuan F., 2016, *ApJ*, 818, 152
- Balbus S. A., 2009, preprint (arXiv:0906.0854)
- Balbus S. A., Hawley J. F., 1991, *ApJ*, 376, 214
- Balbus S. A., Terquem C., 2001, *ApJ*, 552, 235
- Barge P., Sommeria J., 1995, *A&A*, 295, L1
- Barker A. J., Latter H. N., 2015, *MNRAS*, 450, 21
- Béthune W., Lesur G., Ferreira J., 2017, *A&A*, 600, A75
- Birnstiel T., Dullemond C. P., Brauer F., 2010, *A&A*, 513, A79
- Blaes O. M., Balbus S. A., 1994, *ApJ*, 421, 163
- Burns K. J., Vasil G. M., Oishi J. S., Lecoanet D., Brown B. P., 2020, *Phys. Rev. Res.*, 2, 023068
- Chandrasekhar S., 1961, *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford
- Chen K., Lin M.-K., 2020, *ApJ*, 891, 132
- Chiang E., Youdin A. N., 2010, *Annu. Rev. Earth Planet. Sci.*, 38, 493
- Chiang E. I., Goldreich P., 1997, *ApJ*, 490, 368
- Clarke C. J., Pringle J. E., 1988, *MNRAS*, 235, 365
- Cui C., Bai X.-N., 2020, *ApJ*, 891, 30
- Cuzzi J. N., Hogan R. C., Shariff K., 2008, *ApJ*, 687, 1432
- Dubrule B., Morfill G., Sterzik M., 1995, *Icarus*, 114, 237
- Flaherty K. et al., 2020, *ApJ*, 895, 109
- Flaherty K. M., Hughes A. M., Teague R., Simon J. B., Andrews S. M., Wilner D. J., 2018, *ApJ*, 856, 117
- Flaherty K. M. et al., 2017, *ApJ*, 843, 150
- Flock M., Nelson R. P., Turner N. J., Bertrange G. H. M., Carrasco-González C., Henning T., Lyra W., Teague R., 2017, *ApJ*, 850, 131
- Flock M., Turner N. J., Nelson R. P., Lyra W., Manger N., Klahr H., 2020, *ApJ*, 897, 155
- Lizkandra F.-R., Mario F., Riouhei N., 2020, *A&A*, 644, 8
- Fricke K., 1968, *Z. Astrophys.*, 68, 317
- Gammie C. F., 1996, *ApJ*, 457, 355
- Gammie C. F., Balbus S. A., 1994, *MNRAS*, 270, 138
- Goldreich P., Schubert G., 1967, *ApJ*, 150, 571
- Gressel O., Ramsey J. P., Brinch C., Nelson R. P., Turner N. J., Bruderer S., 2020, *ApJ*, 896, 126
- Gressel O., Turner N. J., Nelson R. P., McNally C. P., 2015, *ApJ*, 801, 84
- Johansen A., Klahr H., 2005, *ApJ*, 634, 1353
- Johansen A., Oishi J. S., Mac Low M.-M., Klahr H., Henning T., Youdin A., 2007, *Nature*, 448, 1022
- Johansen A., Youdin A., Mac Low M.-M., 2009, *ApJ*, 704, L75
- Klahr H., Hubbard A., 2014, *ApJ*, 788, 21
- Klahr H. H., Bodenheimer P., 2003, *ApJ*, 582, 869
- Klahr H. H., Henning T., 1997, *Icarus*, 128, 213
- Kunz M. W., 2008, *MNRAS*, 385, 1494
- Kunz M. W., Balbus S. A., 2004, *MNRAS*, 348, 355
- Latter H. N., 2016, *MNRAS*, 455, 2608
- Latter H. N., Papaloizou J., 2018, *MNRAS*, 474, 3110 (LP18)
- Lesur G., 2020, preprint (arXiv:2007.15967)
- Lesur G., Papaloizou J. C. B., 2010, *A&A*, 513, A60
- Lesur G. R. J., Latter H., 2016, *MNRAS*, 462, 4549
- Lin M.-K., 2019, *MNRAS*, 485, 5221
- Lin M.-K., Youdin A. N., 2015, *ApJ*, 811, 17
- Lubow S. H., Pringle J. E., 1993, *ApJ*, 409, 360
- Lyra W., 2014, *ApJ*, 789, 77
- Lyra W., Umurhan O. M., 2019, *PASP*, 131, 072001
- Malygin M. G., Klahr H., Semenov D., Henning T., Dullemond C. P., 2017, *A&A*, 605, A30
- Manger N., Klahr H., 2018, *MNRAS*, 480, 2125
- Manger N., Klahr H., Kley W., Flock M., 2020, *MNRAS*, 499, 1841
- Marcus P. S., Pei S., Jiang C.-H., Barranco J. A., Hassanzadeh P., Lecoanet D., 2015, *ApJ*, 808, 87
- Marcus P. S., Pei S., Jiang C.-H., Hassanzadeh P., 2013, *Phys. Rev. Lett.*, 111, 084501
- Nelson R. P., Gressel O., Umurhan O. M., 2013, *MNRAS*, 435, 2610 (N13)
- Nelson R. P., Papaloizou J. C. B., 2004, *MNRAS*, 350, 849
- Ogilvie G. I., 1997, *MNRAS*, 288, 63
- Ormel C. W., Cuzzi J. N., 2007, *A&A*, 466, 413
- Papaloizou J. C. B., Nelson R. P., Snellgrove M. D., 2004, *MNRAS*, 350, 829
- Perez-Becker D., Chiang E., 2011, *ApJ*, 727, 2
- Pessah M. E., Psaltis D., 2005, *ApJ*, 628, 879
- Petersen M. R., Julien K., Stewart G. R., 2007a, *ApJ*, 658, 1236
- Petersen M. R., Stewart G. R., Julien K., 2007b, *ApJ*, 658, 1252
- Pfeil T., Klahr H., 2019, *ApJ*, 871, 150
- Pfeil T., Klahr H., 2020, preprint (arXiv:2008.11195)
- Raettig N., Klahr H., Lyra W., 2015, *ApJ*, 804, 35
- Richard S., Nelson R. P., Umurhan O. M., 2016, *MNRAS*, 456, 3571
- Sano T., Miyama S. M., 1999, *ApJ*, 515, 776
- Schäfer U., Johansen A., Banerjee R., 2020, *A&A*, 635, A190
- Shakura N. I., Sunyaev R. A., 2009, *A&A*, 500, 33 (special issue 500/01: reprint of 1973, *A&A*, 24, 337)
- Simon J. B., Bai X.-N., Armitage P. J., Stone J. M., Beckwith K., 2013b, *ApJ*, 775, 73
- Simon J. B., Bai X.-N., Stone J. M., Armitage P. J., Beckwith K., 2013a, *ApJ*, 764, 66
- Stoll M. H. R., Kley W., 2014, *A&A*, 572, A77
- Stoll M. H. R., Kley W., 2016, *A&A*, 594, A57
- Tassoul J.-L., 1978, *Theory of Rotating Stars*. Princeton Univ. Press, Princeton, NJ
- Teague R. et al., 2016, *A&A*, 592, A49
- Umurhan O. M., Estrada P. R., Cuzzi J. N., 2020, *ApJ*, 895, 4
- Umurhan O. M., Nelson R. P., Gressel O., 2016a, *A&A*, 586, A33
- Umurhan O. M., Shariff K., Cuzzi J. N., 2016b, *ApJ*, 830, 95
- Urpın V., 2003, *A&A*, 404, 397
- Urpın V., Brandenburg A., 1998, *MNRAS*, 294, 399
- Wardle M., 2007, *Ap&SS*, 311, 35
- Weiss B. P., Bai X.-N., Fu R. R., 2021, *Sci. Adv.*, 7, eaba5967
- Youdin A. N., Goodman J., 2005, *ApJ*, 620, 459
- Youdin A. N., Lithwick Y., 2007, *Icarus*, 192, 588

## APPENDIX A: PROPERTIES OF THE LOCAL DISPERSION RELATION

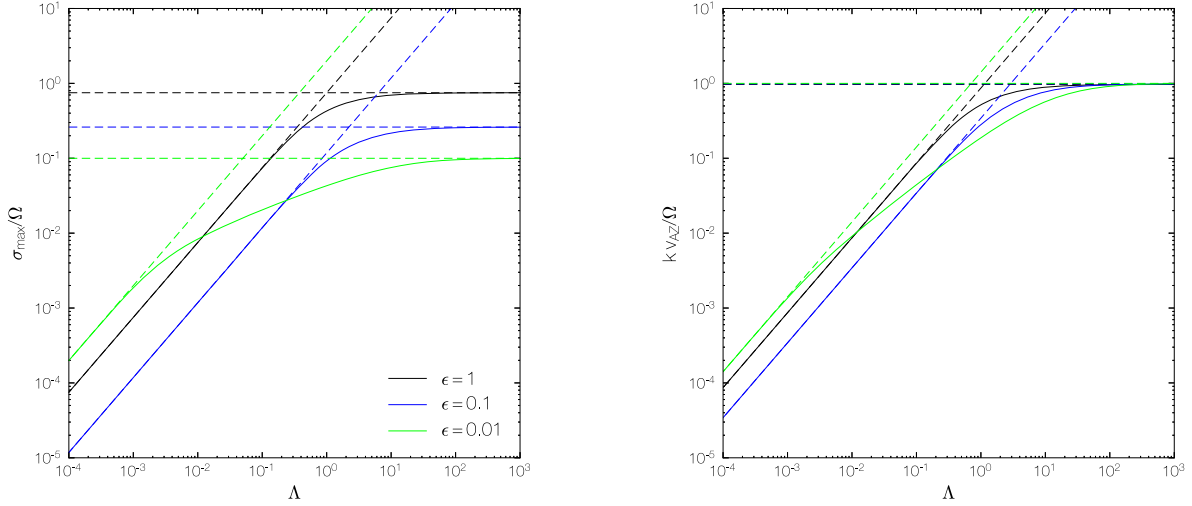
We explore the properties of the local dispersion relation equation (84) in the limit of  $|k_R B_R| \ll |k_Z B_Z|$ . When there is no vertical shear,  $R\partial\Omega^2/\partial Z = 0$ , equation (84) is identical to the dispersion relation for the MRI in equation (22) of Sano & Miyama (1999). When there is no magnetic field,  $v_{AZ} = 0$  and  $\eta = 0$ , our dispersion relation recovers equation (34) of Goldreich & Schubert (1967) in the case when Brunt–Väisälä frequency vanishes. When there is a magnetic field but  $\eta = 0$ , the dispersion relation recovers equation (58) of LP18.

In the hydrodynamic limit,  $v_{AZ} = 0$  and  $\eta = 0$ , the dispersion relation resembles that of pure VSI,

$$\sigma^2 + \frac{k_Z^2}{k^2}(\Omega^2 - A) = 0. \quad (\text{A1})$$

Defining  $\epsilon \equiv k_Z^2/k^2$  and taking  $\partial\Omega/\partial Z < 0$  without loss of generality, we can write  $k_R/k_Z = -(1/\epsilon - 1)^{1/2}$ . The most unstable wavenumbers satisfy

$$\Omega = -\frac{2\epsilon(1/\epsilon - 1) - 1}{\epsilon(1/\epsilon - 1)^{1/2}} \frac{R\partial\Omega}{\partial Z}. \quad (\text{A2})$$



**Figure A1.** Maximum growth rates and most unstable wavenumber are shown as functions of Ohmic Elsässer number  $\Lambda$  by equation (A4) at fixed Alfvén velocity and  $R\partial\Omega^2/\partial Z = -0.1$ . Dashed lines are asymptotic solutions in the limit of  $\eta \rightarrow 0$  (equations A5 and A6) and  $\eta \rightarrow \infty$  (equations A7 and A8).

The vertical shear rate is on the order of  $-R\partial\Omega/\partial Z \sim O(\Omega h)$  via equation (31), hence  $-\Omega/(R\partial\Omega/\partial Z) \sim O(h^{-1})$ , which requires  $\epsilon$  to be small. Therefore, the above equation can be written as

$$h^{-1} \approx \lim_{\epsilon \rightarrow 0} \frac{2\epsilon(1/\epsilon - 1) - 1}{\epsilon(1/\epsilon - 1)^{1/2}} \approx \epsilon^{-1/2}, \quad (\text{A3})$$

so that  $k_Z^2/k^2 \sim h^2$  and the maximum growth rate is  $\sigma_{\max} \sim O(\Omega h)$ , which recovers the results in N13.

The full dispersion relation (84) is implicit in  $\sigma$  and  $k$ . To obtain the maximum growth rate, we take the derivative  $\partial/\partial k$  for each term on the left-hand side and assume  $\epsilon$  to be a constant. This yields a relation between the maximum growth rate  $\sigma_{\max}$  and the most unstable wavenumber  $k$ ,

$$2k^2 = \frac{v_{AZ}^2(-2\sigma^2\epsilon + 3\Omega^2\epsilon^2 + A\epsilon^2) - 2\sigma\eta[\sigma^2 + (\Omega^2 - A)\epsilon]}{v_{AZ}^4\epsilon^2 + 2\eta\sigma v_{AZ}^2\epsilon + \eta^2[\sigma^2 + (\Omega^2 - A)\epsilon]}. \quad (\text{A4})$$

The maximum growth rate is computed numerically by substituting  $k$  in equation (84), and the results are shown in Fig. A1. In the limit

of weak resistivity  $\eta \rightarrow 0$ , the maximum growth rate and the most unstable wavenumber are

$$\sigma_{\max}^2 = \frac{9\Omega^2 + A[(6\Omega^2 + A)/\Omega^2]}{16}\epsilon, \quad (\text{A5})$$

$$k^2 = \frac{15\Omega^2 + A(2\Omega^2 - A)/\Omega^2}{16v_{AZ}^2}. \quad (\text{A6})$$

When there is no vertical shear  $A = 0$ , these expressions recover the MRI channel modes. When  $\eta \rightarrow \infty$ ,

$$\sigma_{\max}^2 = \frac{(3\Omega^2 + A)^2}{16(\Omega^2 - A)^2} \frac{v_{AZ}^4}{\eta^2} \epsilon^2, \quad (\text{A7})$$

$$k^2 = \frac{(3\Omega^2 + A)}{4(\Omega^2 - A)} \frac{v_{AZ}^2}{\eta^2} \epsilon, \quad (\text{A8})$$

and we recover Sano & Miyama (1999) when  $A = 0$ .

This paper has been typeset from a  $\text{\LaTeX}$  file prepared by the author.